$$\frac{S(z)}{E(z)} = H(z) = \frac{1}{1+\sum_{k=1}^{p} a_k z^{-k}} \longrightarrow \boxed{\begin{array}{l} s[n] = e[n] - \sum_{k=1}^{p} a_k s[n-k] \\ e[n] = s[n] + \sum_{k=1}^{p} a_k s[n-k] \end{array}}$$

Let $r[n] = s[n] + \sum_{k=1}^{p} a_k s[n-k]$

$r[n] \neq e[n]$

Find $\{a_k\}$ s.t. $\sum_n r^2[n]$ is min

$$\sum_{k=1}^{p} a_k \sum_n s[n-k]\, s[n-i] = -\sum_n s[n]\, s[n-i], \quad 1 \le i \le p \Rightarrow \boxed{\underset{p\times p}{R}\ \underset{p\times 1}{a} = -\underset{p\times 1}{\rho}}$$

$$E_{p,min} = \sum_n s^2[n] + \sum_{k=1}^{p} a_k \sum_n s[n]\, s[n-k]$$

- stationary / Autocorrelation formulation (ANE)

$$\sum_{n=-\infty}^{\infty} r^2[n] \Rightarrow \sum_{k=1}^{p} a_k R_{xx}[|k-i|] = -R_{xx}[i], \quad 1 \le i \le p$$

$$E_{p,min} = R_{xx}[0] + \sum_{k=1}^{p} a_k R_{xx}[k]$$

- Non-stationary / Covariance formulation (CNE)

$$\sum_{n=0}^{N-1} r^2[n] \Rightarrow \sum_{k=1}^{p} a_k \phi_{ki} = -\phi_{0i}, \quad 1 \le i \le p$$

$$E_{p,min} = \phi_{00} + \sum_{k=1}^{p} a_k \phi_{0k}$$

* $R[i,j] = R[i+1,j+1] \Rightarrow R_{xx}$ toeplitz symm
* $\phi_{ij} \neq \phi_{i+1,j+1} = \sum_{n=0}^{N-1} s[n-1-i]\, s[n-j-1]$
$= \phi_{ij} + s[-i-1]\, s[-j-1] - s[N-1-j]\, s[N-1-i]$

$\Rightarrow \phi$ symm. but NOT toeplitz

$$H(z) = \frac{G}{A(z)}, \text{ causal}$$

$$= \frac{G}{1+\sum_{k=1}^{p} a_k z^{-k}}$$

$\Rightarrow$

$$R_{hh}[0] = G^2 - \sum_{k=1}^{p} a_k R_{hh}[k]$$

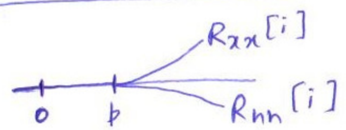
$$R_{hh}[i] = -\sum_{k=1}^{p} a_k R_{hh}[k-i], \quad i > 0$$

Goal: Power match

$R_{hh}[0] = R_{xx}[0]$

Comparing with ANE,

$$\boxed{\begin{array}{l} E_p = G^2 \\ R_{xx}[i] = R_{hh}[i], \quad 1 \le i \le p \end{array}}$$



$$\begin{bmatrix} R(0) & R(1) & \cdots & R(p-1) \\ R(1) & R(0) & \cdots & R(p-2) \\ \vdots & \vdots & & \vdots \\ R(p-1) & R(p-2) & \cdots & R(0) \end{bmatrix}_{p\times p} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}_{p\times 1} = - \begin{bmatrix} R(1) \\ R(2) \\ \vdots \\ R(p) \end{bmatrix}_{p\times 1} ; \qquad R(0) + \sum_{k=1}^{p} a_k R(k) = E_p$$

Toeplitz
symm

$$\begin{bmatrix} R(0) & R(1) & R(2) & \cdots & R(p) \\ R(1) & R(0) & R(1) & \cdots & R(p-1) \\ R(2) & R(1) & R(0) & \cdots & R(p-2) \\ \vdots & \vdots & \vdots & & \vdots \\ R(p) & R(p-1) & R(p-2) & \cdots & R(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} E_p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underline{a}_p = [1 \quad a_1^p \quad a_2^p \cdots \quad a_p^p]^T$$

$p=1$

$$\begin{bmatrix} R(0) & R(1) \\ R(1) & R(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1^1 \end{bmatrix} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad a_1^1 = -\frac{R(1)}{R(0)}$$

$$E_1 = R(0) + a_1^1 R(1)$$

$$\underline{a}_p = [1 \quad a_1^p \quad a_2^p \quad \cdots \quad a_p^p]^T$$

$p=2$

$$\begin{bmatrix} R(0) & R(1) & R(2) \\ R(1) & R(0) & R(1) \\ R(2) & R(1) & R(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1^2 \\ a_2^2 \end{bmatrix} = \begin{bmatrix} E_2 \\ 0 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} R(0) & R(1) & R(2) \\ R(1) & R(0) & R(1) \\ R(2) & R(1) & R(0) \end{bmatrix} \begin{bmatrix} a_2^2 \\ a_1^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_2 \end{bmatrix}$$

$p=3$

$$\begin{bmatrix} R(0) & R(1) & R(2) & R(3) \\ R(1) & R(0) & R(1) & R(2) \\ R(2) & R(1) & R(0) & R(1) \\ R(3) & R(2) & R(1) & R(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1^3 \\ a_2^3 \\ a_3^3 \end{bmatrix} = \begin{bmatrix} E_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

?

$$\begin{bmatrix} 1 \\ a_1^2 \\ a_2^2 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ a_2^2 \\ a_1^2 \\ 1 \end{bmatrix} = \begin{bmatrix} E_2 \\ 0 \\ 0 \\ q \end{bmatrix} + k_3 \begin{bmatrix} q \\ 0 \\ 0 \\ E_2 \end{bmatrix}$$

$$q = \sum_{k=0}^{2} a_k^2 R(3-k)$$

$$q + k_3 E_2 = 0 \Rightarrow k_3 = \frac{-q}{E_2}$$

$$\begin{aligned} E_3 &= E_2 + k_3 q \\ &= E_2 + k_3(-k_3 E_2) \\ &= (1 - k_3^2) E_2 \end{aligned}$$

$$|k_3| \overset{?}{\leq} 1$$

Lenvinson - Durbin Algo

- $E_0 = R(0)$, $a_0^0 = 1$
- for $1 \le i \le p$.

$$k_i = \frac{-\sum_{k=0}^{i-1} a_k^{i-1} R(i-k)}{E_{i-1}}$$

for $0 \le j \le i$

$$a_j^i = a_j^{i-1} + k_i a_{i-j}^{i-1} ,$$

end for

$$E_i = (1 - k_i^2) E_{i-1}$$

end for

$$\underline{a}^p \to 1 \quad a_1^p \quad a_2^p \quad \cdots \quad a_{p-1}^p \quad a_p^p$$

$$\underline{b}^p \to a_p^p \quad a_{p-1}^p \quad \cdots \quad a_2^p \quad a_1^p \quad 1$$

$$A_p(z) = 1 + a_1^p z^{-1} + a_2^p z^{-2} + \cdots + a_{p-1}^p z^{-(p-1)} + a_p^p z^{-p}$$

$$A_p(1/z) = 1 + a_1^p z + a_2^p z^2 + \cdots + a_{p-1}^p z^{(p-1)} + a_p^p z^p$$

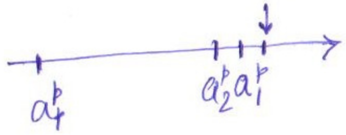
$$B_p(z) = a_p^p + a_{p-1}^p z^{-1} + \cdots + a_2^p z^{-(p-2)} + a_1^p z^{-(p-1)} + z^{-p} \Rightarrow z^{-p} A_p(1/z) = B_p(z)$$

$x[n] \to \boxed{A_p(z)} \to e_p^+[n]$

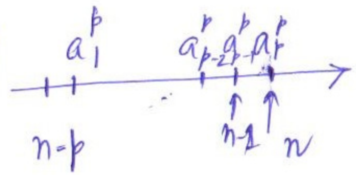
$x[n] \to \boxed{B_p(z)} \to e_p^-[n]$

$$\boxed{\begin{aligned} A_p(z) &= A_{p-1}(z) + k_p z^{-1} B_{p-1}(z) \\ &= A_{p-1}(z) + k_p z^{-1} z^{-(p-1)} A_{p-1}(1/z) \\ A_p(z) &= A_{p-1}(z) + k_p z^{-p} A_{p-1}(1/z) \end{aligned}}$$

FORWARD
$$\begin{cases} e_p^+(z) = X(z) A_p(z) = X(z)\left(1 + \sum_{k=1}^{p} a_k^p z^{-k}\right) \\ e_p^+[n] = x[n] + \sum_{k=1}^{p} a_k^p x[n-k] \end{cases}$$

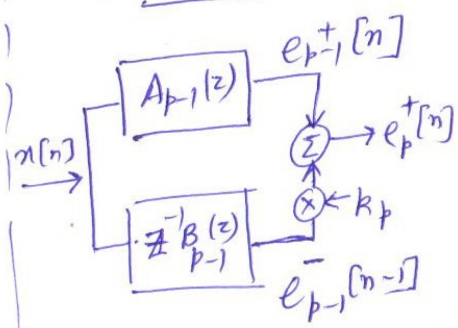


BACKWARD
$$\begin{cases} e_p^-(z) = X(z) B_p(z) = X(z) z^{-p} A_p(1/z) \\ \qquad\quad = X(z) z^{-p}\left(1 + \sum_{k=1}^{p} a_k^p z^{k}\right) \\ e_p^-[n] = x[n-p] + \sum_{k=1}^{p} a_k^p x[n+k-p] \end{cases}$$



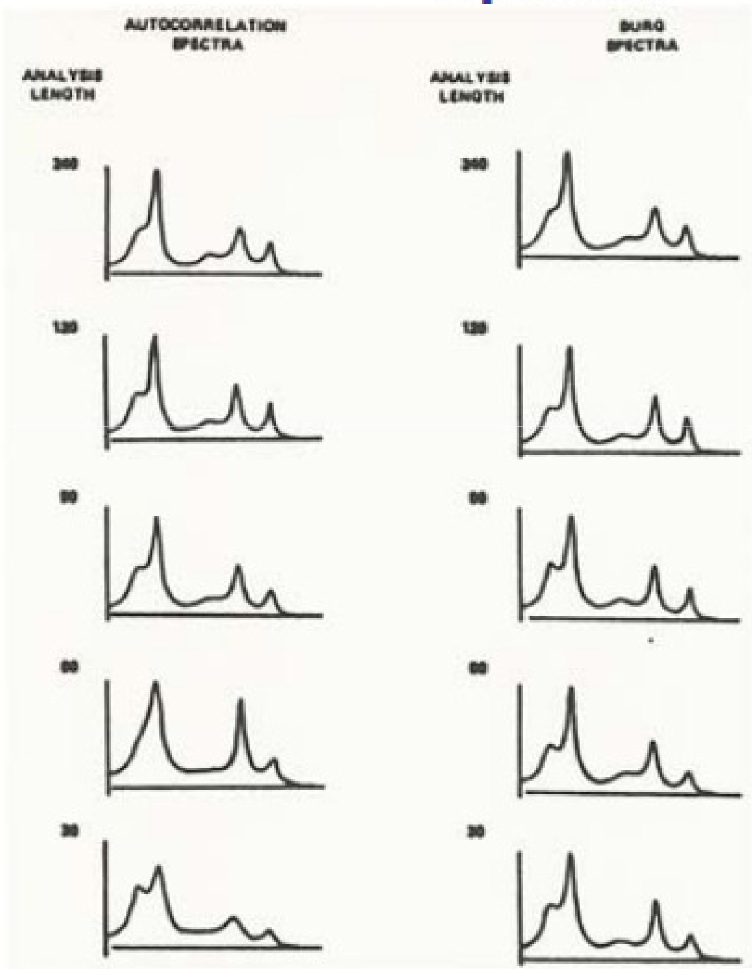
$e_p^+[n] \neq e_p^-[n]$ but $\{a_k^p\}$ are same & $E_p^+ = E_p^-$

$x[n] \to \boxed{A_p(z)} \to e_p^+[n]$



$$e_p^+[n] = e_{p-1}^+[n] + k_p e_{p-1}^-[n-1]$$

Comparison of Autocorrelation and Burg Spectra



- significantly less smearing of formant peaks using Burg method

$$\boxed{A_p(z) = A_{p-1}(z) + k_p z^{-1} B_{p-1}(z)}$$

$$z^{-p} A_p(1/z) = B_p(z)$$
$$\Downarrow$$
$$z^{p} A_p(z) = B_p(1/z)$$
$$\Downarrow$$
$$A_p(1/z) = z^{p} B_p(z)$$

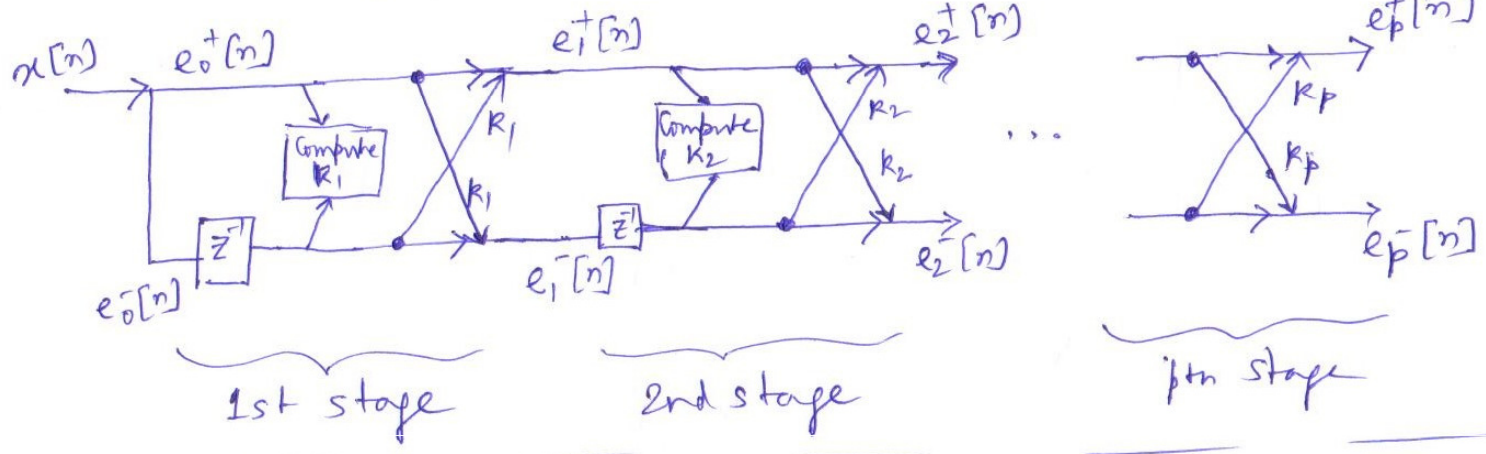
$$
\begin{aligned}
B_p(z) &= z^{-p} A_p(1/z) \\
&= z^{-p}\left\{ A_{p-1}(z^{-1}) + k_p z^{+1} B_{p-1}(z^{-1}) \right\} \\
&= z^{-p} A_{p-1}(1/z) + k_p z^{-(p-1)} \underbrace{B_{p-1}(1/z)} \\
&= z^{-1} \underbrace{z^{-(p-1)} A_{p-1}(1/z)} + k_p z^{-(p-1)} z^{p-1} A_{p-1}(z)
\end{aligned}
$$

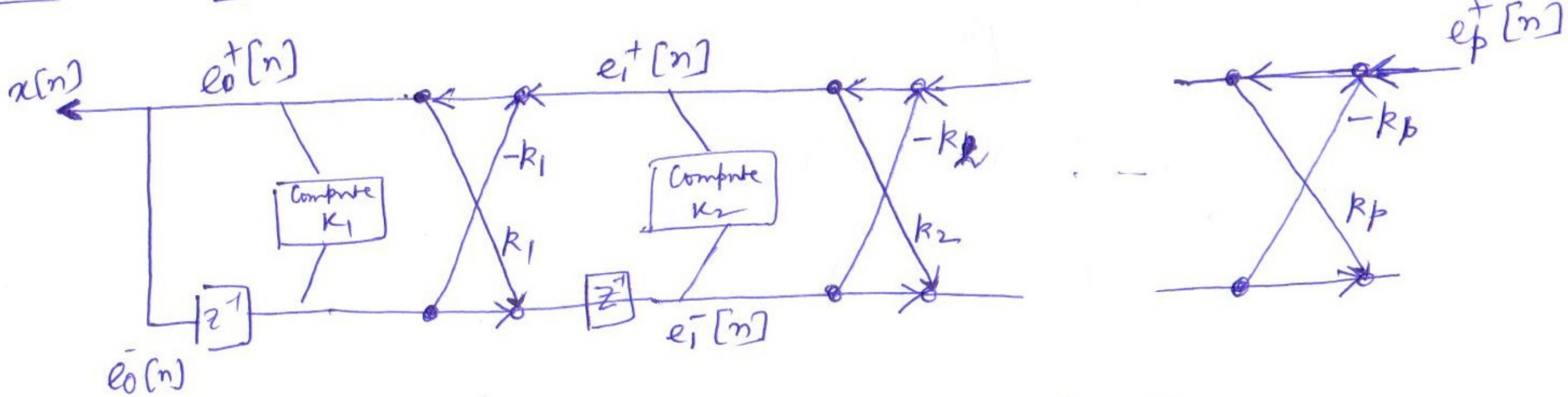
$$\boxed{B_p(z) = z^{-1} B_{p-1}(z) + k_p A_{p-1}(z)}$$

$$
\begin{bmatrix} A_i(z) \\ B_i(z) \end{bmatrix} = \begin{bmatrix} 1 & k_i z^{-1} \\ k_i & z^{-1} \end{bmatrix} \begin{bmatrix} A_{i-1}(z) \\ B_{i-1}(z) \end{bmatrix}
$$

$$e_i^+[n] = e_{i-1}^+[n] + k_i\, e_{i-1}^-[n-1]$$

$$e_i^-[n] = e_{i-1}^-[n-1] + k_i\, e_{i-1}^+[n]$$





$$e_{i-1}^+[n] = e_i^+[n] - k_i\, e_{i-1}^-[n-1]$$

Stability of $\dfrac{1}{A_i(z)} = \dfrac{1}{\prod_{k=1}^{M_1}(1-\alpha_k z^{-1}) \prod_{k=1}^{M_2}(1-z_k z^{-1})(1-z_k^* z^{-1})} = \dfrac{1}{1+\sum_{k=1}^{i} a_k^i z^{-k}}$

$$\boxed{1/A_i(z) \text{ stable} \iff |k_i| < 1}$$

$a_i^i = k_i = \prod_{k=1}^{M_1} \alpha_k \prod_{k=1}^{M_2} r_k^2$, where $z_k = r_k e^{j\theta_k}$

if $\frac{1}{A_i(z)}$ is stable $\Rightarrow |\alpha_k|, |z_k| = r_k < 1 \Rightarrow |k_i| < 1$.

$A_i(z) = A_{i-1}(z) + k_i z^{-1} B_{i-1}(z) \Rightarrow A_{i-1}(z) = A_i(z) - k_i z^{-1} B_{i-1}(z)$

$B_i(z) = z^{-i} A_i(1/z) = z^{-1} B_{i-1}(z) + k_i A_{i-1}(z) \Rightarrow B_{i-1}(z) = z\left[B_i(z) - k_i A_{i-1}(z)\right]$

$A_{i-1}(z) = A_i(z) - k_i z^{-1}$

$\Rightarrow A_{i-1}(z) = \dfrac{A_i(z) - k_i B_i(z)}{(1-k_i^2)}$

at $z_{i,l}$, $A_i(z_{i,l}) = 0$

$$\boxed{k_i = \frac{-A_{i-1}(z_{i,l})}{z_{i,l}^{-1} B_{i-1}(z_{i,l})} = -z X_{i-1}(z)\Big|_{z_{i,l}}}$$

at $z_{i-1,l}$

$$\boxed{k_i = \frac{A_i(z_{i-1,l})}{B_i(z_{i-1,l})} = X_i(z)\Big|_{z_{i-1,l}}}$$

* $z_{i,l}$: l^{th} root of i^{th} order polynomial
* $X_i(z) \triangleq A_i(z)/B_i(z)$: $|X_i(e^{j\omega})| = 1$ & $\begin{cases} =1, & |z|=1 \\ >1, & |z|>1 \\ <1, & |z|<1 \end{cases}$

If $|z_{i-1,l}| < 1$, $|k_i| < 1 \Rightarrow |z_{i,l} X_{i-1}(z_{i,l})| < 1 \Rightarrow |z_{i,l}|\,|X_{i-1}(z_{i,l})| < 1$

$\Rightarrow |z_{i,l}| < 1$