$$\frac{S(z)}{E(z)} = H(z) = \frac{1}{1+\sum_{k=1}^{p} a_k z^{-k}} \longrightarrow \boxed{\begin{array}{l} s[n] = e[n] - \sum_{k=1}^{p} a_k s[n-k] \\ e[n] = s[n] + \sum_{k=1}^{p} a_k s[n-k] \end{array}}$$

Let $r[n] = s[n] + \sum_{k=1}^{p} a_k s[n-k]$

$\quad\; \# \; e[n]$

find $\{a_k\}$ s.t. $\sum_n r^2[n]$ is min

$$\sum_{k=1}^{p} a_k \sum_n s[n-k]\, s[n-i] = -\sum_n s[n]\, s[n-i], \quad 1 \le i \le p \Rightarrow \boxed{\underset{p\times p}{R}\; \underset{p\times 1}{a} = -\underset{p\times 1}{\rho}}$$

$$E_{p,\min} = \sum_n s^2[n] + \sum_{k=1}^{p} a_k \sum_n s[n]\, s[n-k]$$

$$H(z) = \frac{G}{A(z)}, \text{ causal}$$

$$= \frac{G}{1+\sum_{k=1}^{p} a_k z^{-k}}$$

$$\Rightarrow R_{hh}[0] = G^2 - \sum_{k=1}^{p} a_k R_{hh}[k]$$

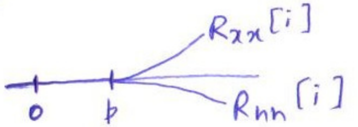
$$R_{hh}[i] = -\sum_{k=1}^{p} a_k R_{hh}[k-i], \quad i > 0$$

Goal: Power match

$R_{hh}[0] = R_{xx}[0]$

Comparing with ANE,

$$\boxed{\begin{array}{l} E_p = G^2 \\ R_{xx}[i] = R_{hh}[i], \; 1 \le i \le p \end{array}}$$



- stationary / Autocorrelation formulation (ANE)

$$\sum_{n=-\infty}^{\infty} r^2[n] \Rightarrow \sum_{k=1}^{p} a_k R_{xx}[|k-i|] = -R_{xx}[i], \; 1 \le i \le p$$

$$E_{p,\min} = R_{xx}[0] + \sum_{k=1}^{p} a_k R_{xx}[k]$$

- Non-stationary / Covariance formulation (CNE)

$$\sum_{n=0}^{N-1} r^2[n] \Rightarrow \sum_{k=1}^{p} a_k \phi_{ki} = -\phi_{0i}, \; 1 \le i \le p$$

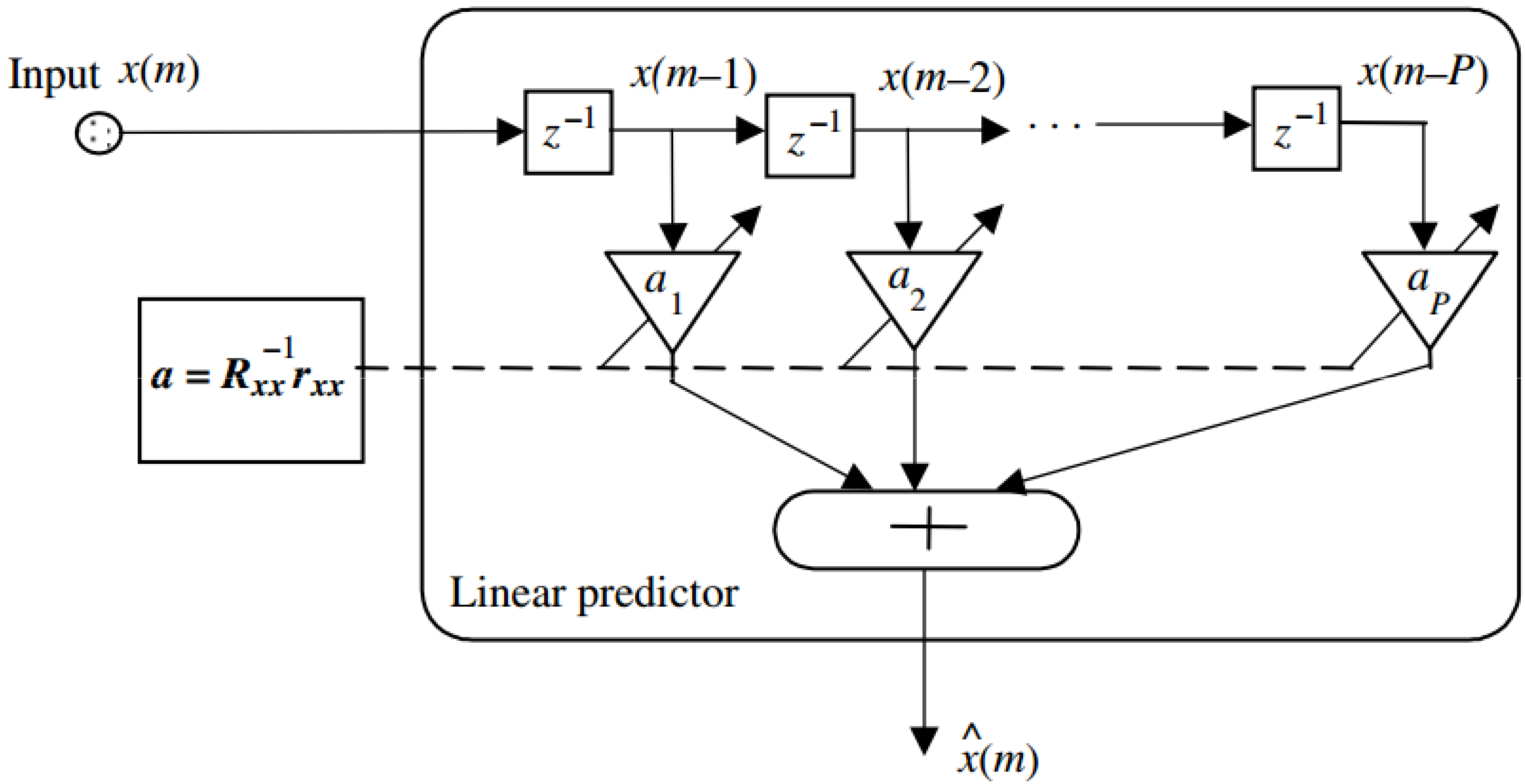
$$E_{p,\min} = \phi_{00} + \sum_{k=1}^{p} a_k \phi_{0k}$$

$*$ $R[i,j] = R[i+1, j+1] \Rightarrow R_{xx}$ toeplitz symm

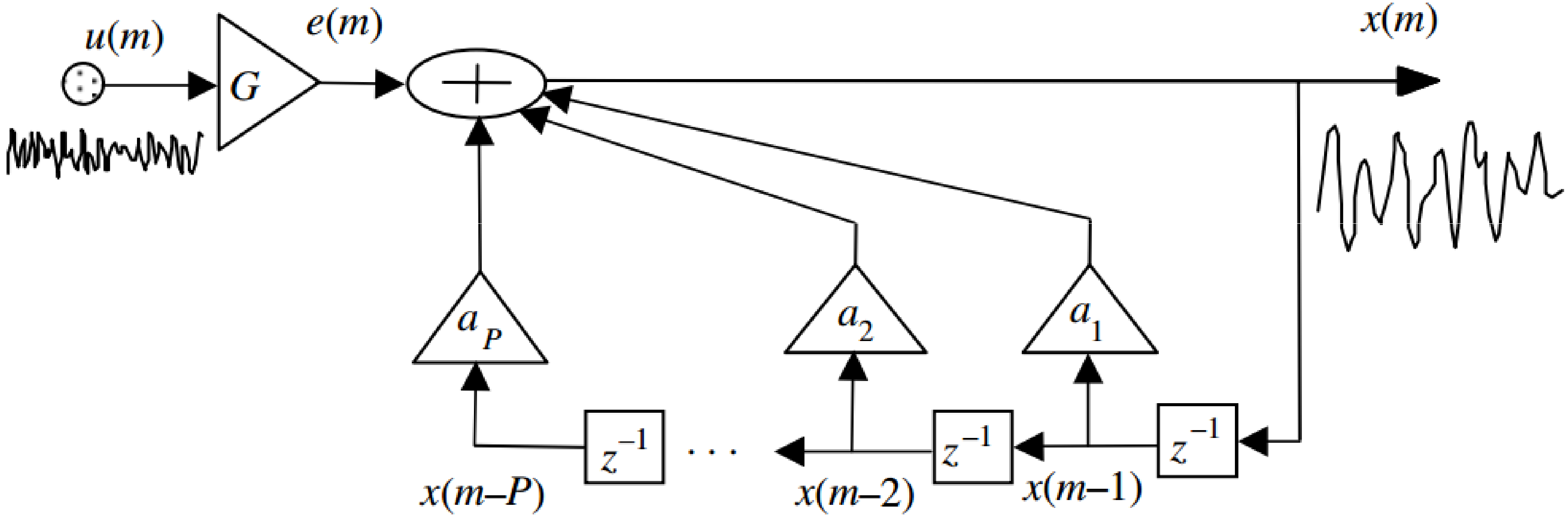
$$* \; \phi_{ij} \ne \phi_{i+1,j+1} = \sum_{n=0}^{N-1} s[n-1-i]\, s[n-j-1]$$

$$= \phi_{ij} + s[-i-1]\, s[-j-1] - s[N-1-j]\, s[N-1-i]$$

$\Rightarrow \phi$ symm. but NOT toeplitz



Input $x(m)$
$x(m-1)$
$x(m-2)$
$x(m-P)$
z^{-1}
z^{-1}
z^{-1}
a_1
a_2
a_P
$\boldsymbol{a} = \boldsymbol{R}_{xx}^{-1}\boldsymbol{r}_{xx}$
Linear predictor
$\hat{x}(m)$



u(m)
G
e(m)
x(m)
a_P
a_2
a_1
z^-1
z^-1
z^-1
x(m–P)
. . .
x(m–2)
x(m–1)

$$\underbrace{\begin{bmatrix} R(0) & R(1) & \cdots & R(p-1) \\ R(1) & R(0) & \cdots & R(p-2) \\ \vdots & \vdots & & \vdots \\ R(p-1) & R(p-2) & \cdots & R(0) \end{bmatrix}}_{p\times p} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}}_{p\times 1} = -\underbrace{\begin{bmatrix} R(1) \\ R(2) \\ \vdots \\ R(p) \end{bmatrix}}_{p\times 1} ; \qquad R(0) + \sum_{k=1}^{p} a_k R(k) = E_p$$

Toeplitz
symm

$$\begin{bmatrix} R(0) & R(1) & R(2) & \cdots & R(p) \\ R(1) & R(0) & R(1) & \cdots & R(p-1) \\ R(2) & R(1) & R(0) & \cdots & R(p-2) \\ \vdots & \vdots & \vdots & & \vdots \\ R(p) & R(p-1) & R(p-2) & \cdots & R(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} E_p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underline{a}_p = [1 \quad a_1^p \quad a_2^p \cdots \quad a_p^p]^T$$

$p=1$

$$\begin{bmatrix} R(0) & R(1) \\ R(1) & R(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1^1 \end{bmatrix} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad a_1^1 = -\frac{R(1)}{R(0)}$$

$$E_1 = R(0) + a_1^1 R(1)$$

$$\underline{a}_p = [1 \quad a_1^p \quad a_2^p \quad \cdots \quad a_p^p]^T$$

$p=2$

$$\begin{bmatrix} R(0) & R(1) & R(2) \\ R(1) & R(0) & R(1) \\ R(2) & R(1) & R(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1^2 \\ a_2^2 \end{bmatrix} = \begin{bmatrix} E_2 \\ 0 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} R(0) & R(1) & R(2) \\ R(1) & R(0) & R(1) \\ R(2) & R(1) & R(0) \end{bmatrix} \begin{bmatrix} a_2^2 \\ a_1^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_2 \end{bmatrix}$$

$p=3$

$$\begin{bmatrix} R(0) & R(1) & R(2) & R(3) \\ R(1) & R(0) & R(1) & R(2) \\ R(2) & R(1) & R(0) & R(1) \\ R(3) & R(2) & R(1) & R(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1^3 \\ a_2^3 \\ a_3^3 \end{bmatrix} = \begin{bmatrix} E_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Downarrow ?$

$$\begin{bmatrix} 1 \\ a_1^2 \\ a_2^2 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ a_2^2 \\ a_1^2 \\ 1 \end{bmatrix} = \begin{bmatrix} E_2 \\ 0 \\ 0 \\ q \end{bmatrix} + k_3 \begin{bmatrix} q \\ 0 \\ 0 \\ E_2 \end{bmatrix}$$

$$q = \sum_{k=0}^{2} a_k^2 R(3-k)$$

$$q + k_3 E_2 = 0 \Rightarrow k_3 = \frac{-q}{E_2}$$

$$\begin{aligned} E_3 &= E_2 + k_3 q \\ &= E_2 + k_3(-k_3 E_2) \\ &= (1 - k_3^2) E_2 \end{aligned}$$

$|k_3| \overset{?}{<} 1$

Lenvinson - Durbin Algo

- $E_0 = R(0)$, $a_0^0 = 1$
- for $1 \le i \le p$.

$$k_i = \frac{-\sum_{k=0}^{i-1} a_k^{i-1} R(i-k)}{E_{i-1}}$$

for $0 \le j \le i$

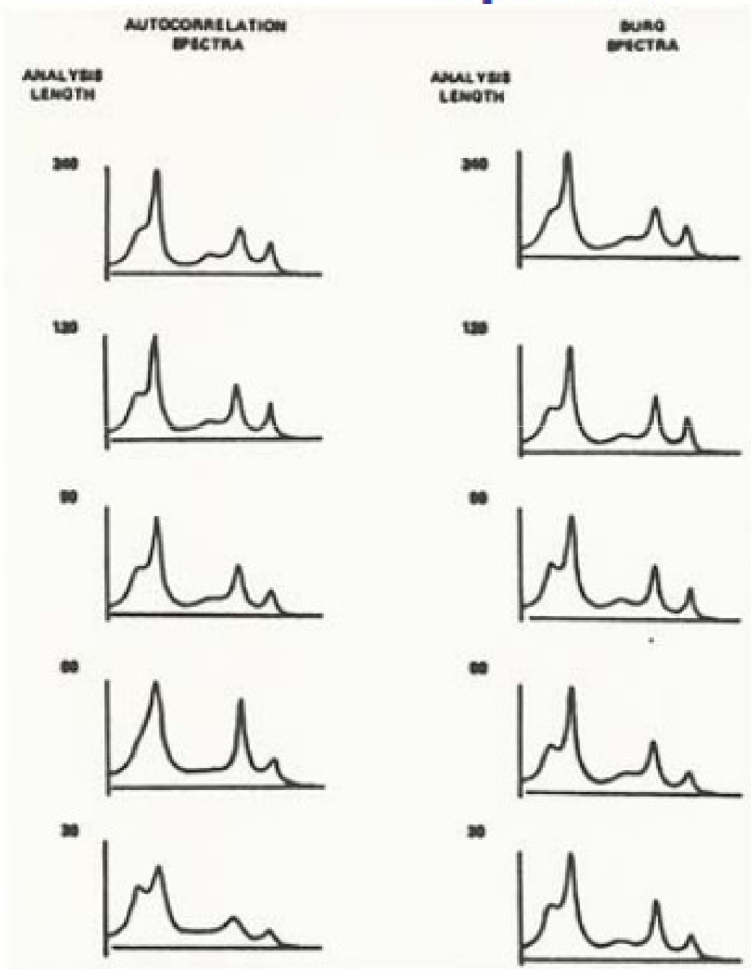
$$a_j^i = a_j^{i-1} + k_i \, a_{i-j}^{i-1} ,$$

end for

$$E_i = (1 - k_i^2) E_{i-1}$$

end for

Comparison of Autocorrelation and Burg Spectra



- significantly less smearing of formant peaks using Burg method