When is a Periodic Discrete-Time System Equivalent to a Time-Invariant One?

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ABSTRACT

We give the precise conditions under which a periodic discrete-time linear state-space system can be transformed into a time-invariant one by a change of basis. Thus our theory is the discrete-time counterpart of the classical theory of Floquet transforms developed by Floquet and Lyapunov in the 1800s for continuous-time systems. We state and prove a necessary and sufficient condition for a "discrete-time Floquet transform" to exist, and give a construction for the transform when it does exist. Our results also extend to generalized state-space, or descriptor, systems.

1. INTRODUCTION

Perhaps the simplest class of time-varying linear systems is that for which the time variations are periodic. Even so, such systems exhibit significant departure in behavior from time-invariant ones and continue to be a topic of intensive study. In the state-space formulation of linear systems, periodicity implies certain structural properties of the associated transition matrix, and it is often possible to reduce the study of the original periodic system to that of a time-invariant one. This aspect of periodic systems was studied by Floquet and Lyapunov in the 1800s for the continuous-time case, and their result is generally referred to as the *Floquet transform* method [7, 9]. However, there are no corresponding results for the discrete-time case,

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and the question as to when a periodic discrete-time linear system is equivalent to a time-invariant one has remained open so far. In this paper, we completely solve this problem, and give a construction for a discrete-time analogue (when it exists) of the Floquet transform.

For completeness, and also to motivate our problem, we shall briefly discuss continuous-time Floquet theory first. The Floquet transform is a classical tool in the study of linear differential equations, and is elegantly described in terms of the following theorem:

THEOREM 1 (Floquet-Lyapunov). *Given the homogeneous linear state-space system*

$$\dot{x}(t) = A(t)x(t), \qquad x(0) = x_0, \tag{1}$$

where $A(t) \in \mathcal{R}^{n \times n}$ *is bounded and at least piecewise continuous in* $(-\infty, +\infty)$ *with period* T,

$$A(t) = A(t+T) \quad \forall t \qquad (T \text{ minimal}),$$

there exists a matrix $P(t)$, *nonsingular for all* t, *and periodic of period* T *with* $P(0) = I$, *such that the change of variables* $x(t) = P(t)\widehat{x}(t)$ *transforms the system into a linear system with constant coefficients.*

Since discrete-time and continuous-time Floquet transforms are somewhat related, we sketch here the main ideas of a proof of Theorem 1. The interested reader is referred to [7] or [9] for details. Let $\Phi(t_2, t_1)$ denote the state transition matrix of the system (1); then the state transition matrix over one period is, $\Phi(T, 0)$, called the *monodromy matrix*. Define a constant matrix R such that

$$\Phi(T, 0) = e^{RT}. \tag{2}$$

This is always possible, since any nonsingular matrix can be expressed as an exponential [3]. The matrix R is in some sense a *logarithm* of $\Phi(T, 0)$, up to a factor T. It is complex in general. Then define

$$P(t) = \Phi(t, 0)e^{-Rt}. \tag{3}$$

It is easy to verify that $P(0) = I$, and that $P(t)$ is nonsingular and periodic of period T. Performing the change of variables $x(t) = P(t)\widehat{x}(t)$ in (1), we see that $\widehat{x}(t)$ satisfies

$$\dot{\widehat{x}}(t) = P^{-1}(AP - \dot{P})\widehat{x}(t), \qquad \widehat{x}(0) = P^{-1}(0)x_0 = x_0. \tag{4}$$

$$\begin{tikzcd} C^n \arrow[r, "\Phi(t,t_0)"] \arrow[d, "P^{-1}(t_0)"'] & C^n \\ C^n \arrow[r, "e^{R(t-t_0)}"'] & C^n \arrow[u, "P(t)"'] \end{tikzcd}$$

FIG. 1. The Floquet-Lyapunov transformation for periodic differential equations. We have $\Phi(t,\ t_0) = P(t)e^{R(t-t_0)}P^{-1}(t_0)$.

Since from (3),

$$\begin{aligned} \dot{P} &= A(t)\Phi(t,0)e^{-Rt} - \Phi(t,0)e^{-Rt}R \\ &= AP - PR, \end{aligned}$$

Equation (4) reduces to

$$\dot{\widehat{x}}(t) = R\widehat{x}(t), \qquad \widehat{x}(0) = x_0. \tag{5}$$

Thus $\widehat{x}(t) = P^{-1}(t)x(t)$ is the solution of a linear system with constant coefficients. The periodicity of the original system has been absorbed in the transformation $P(t)$—this is shown pictorially in Figure 1.

From the foregoing, we deduce that any linear system with periodic coefficients such as (1) can be viewed as a periodic transformation of a time-invariant system (5). Also, the whole behavior of its solutions depends upon the eigenvalues of the matrix R. These eigenvalues are of the form $(1/T)\ln\lambda_j$, where λ_j are the eigenvalues of the monodromy matrix $\Phi(T,0)$.

Consider now the discrete-time homogeneous state-space system

$$x(j+1) = A_j x(j), \qquad x(0) = x_0, \tag{6}$$

with $A_j = A_{j+K}\ \forall j$ (K minimal). As in the continuous-time case, we expect to gain insight into the structure of the solutions of a linear system with periodic coefficients by transformation into an equivalent time-invariant system [6, 10]. Accordingly, for the system (6), we would like to find an invertible state transformation $T(j)$, periodic of period K, such that the new state vector

$$\widehat{x}(j) := T^{-1}(j)x(j) \tag{7}$$

satisfies

$$\widehat{x}(j+1) = \widehat{A}_j\widehat{x}(j), \qquad \widehat{x}(0) = T^{-1}(0)x_0, \tag{8}$$

where

$$\widehat{A}_j = T^{-1}(j+1)A_jT(j) \equiv \widehat{A} \tag{9}$$

is constant. Notice that $\widehat{A}_j$ is periodic, since $T(j)$ and A_j are, but $T(j)$ has to be chosen so that, furthermore, $\widehat{A}_j$ is now constant, or independent of j. It turns out that the required discrete-time Floquet transform $T(j)$ may not always exist, unlike in the continuous-time case. In Section 2, we give simple examples for which it is impossible to find a discrete-time Floquet transform.

In this paper, specifically Section 3, we derive necessary and sufficient conditions on the system (6) for the existence of invertible matrices $T(j)$ satisfying (9)—so that the periodic state transformation (7) applied to the periodic system (6) leads to the time-invariant system (8). We also give a construction for the discrete-time Floquet transform when it does exist. We use complex arithmetic throughout; however, we do indicate the modifications needed for the real-arithmetic case in Section 4.

2. BACKGROUND

Let us examine what is entailed in finding a discrete-time Floquet transform. By invertible transformations, as shown in (9), all A_j are to be made equal to the same matrix $\widehat{A}$. Writing out Equation (9) for $j = 1, 2, \ldots, K$, and imposing periodicity of $T(j)$, we see that we are required to solve

$$\begin{aligned} T(2)\widehat{A} &= A_1T(1), \\ T(3)\widehat{A} &= A_2T(2), \\ &\vdots \\ T(K)\widehat{A} &= A_{K-1}T(K-1), \\ T(1)\widehat{A} &= A_KT(K) \end{aligned} \tag{10}$$

for $\widehat{A}, T(1), \ldots, T(K)$. This represents K $n \times n$ matrix equations in $K+1$ $n \times n$ matrix unknowns. Moreover, the invertibility conditions amount to

K scalar inequalities in the elements of the matrices $T(j)$. However, we can choose one $T(\cdot)$ arbitrarily, since the matrix $\widehat{A}$ is defined only up to some *constant* similarity transformation. Notice that we can always perform a constant transformation $\widehat{x}(j) = G\overline{x}(j)$, $\det G \neq 0$, on the time-invariant system (8) to obtain a new time-invariant system involving $\overline{x}(j)$. In that case, $\widehat{A}$ would change to $G^{-1}\widehat{A}G$, and $T(j)$ to $T(j)G$, as is evident from an inspection of (9). Hence there is no loss of generality in assuming that, say, $T(1) = I$. This would in fact be obtained by making the changes $\widehat{A} \leftarrow T(1)\widehat{A}T^{-1}(1)$ and $T(j) \leftarrow T(j)T^{-1}(1)$, $1 \leq j \leq K$, in (10). Then (10) consists of K matrix equations in K unknown matrices.

Assuming $T(1) = I$ and multiplying out (10) yields

$$\widehat{A}^K = A_K \cdots A_2 A_1, \tag{11}$$

which can be solved for $\widehat{A}$. We refer to Gantmacher [3, Chapter VIII] for details about the theory of Kth roots of a given matrix. Once $\widehat{A}$ is obtained, we can find $T(2), \ldots, T(K)$ using (10). This is written succinctly below as

$$T(j)\widehat{A} = A_{j-1}T(j-1), \qquad j = 2, \ldots, K. \tag{12}$$

Note that $T(j-1)$ is already known when solving (12) for $T(j)$.

This procedure essentially solves the problem when the *monodromy matrix* $\Phi(K+1, 1) := A_K \cdots A_2 A_1$ is invertible, or equivalently, the system (6) is *reversible*:

$$\det A_j \neq 0 \qquad \forall j,$$

since then a Kth root $\widehat{A}$ always exists and is invertible [3]. For the same reason, the matrices $T(j)$ constructed from (12) are guaranteed to be invertible. The number of solutions equals the number of Kth roots of $\Phi(K+1, 1)$, which is K^n in the generic case. As a check for verifying the above steps, note that (9) implies $T(K+1)\widehat{A}^K = A_K \cdots A_2 A_1 T(1) = A_K \cdots A_2 A_1$, and therefore (11) yields $T(K+1) = I = T(1)$. Periodicity of $T(j)$ is thus satisfied—in fact, we imposed it to obtain Equation (10).

REMARK 1. We draw the reader's attention to the similarity here with the continuous-time approach. In both cases, the monodromy matrix plays a key role—the transformation is derived here from its Kth root, whereas in the continuous-time case, its logarithm has to be taken. In both cases, the solution is not unique or real-valued in general. Furthermore, we have $T(1) = I$ in the discrete-time case, corresponding to $P(0) = I$ in the continuous-time case. However, we shall see that there are a number of important differences between the two cases, especially when the system (6) is *not* reversible.

REMARK 2. Reversible discrete-time systems arise commonly in practice, for instance by discretization of continuous-time systems. There the state vectors $x(j)$ are obtained by sampling a continuous-time state process at instants t_j, and

$$A_j := \Phi(t_{j+1}, t_j),$$

where $\Phi(\cdot,\cdot)$ is the state transition matrix of the continuous-time system. Since a continuous-time state transition matrix is invertible over any interval, the matrices A_j generated in this manner are nonsingular, and we can apply the procedure described above to find a discrete-time Floquet transform.

REMARK 3. Solving (11) and (12) numerically is not a trivial task in general. As with most "straightforward" procedures, the above algorithm may suffer from severe numerical difficulties. It involves the computation of a Kth root of $\Phi(K+1,1)$, which is a dense matrix; and the solution described in [3] requires not only the eigenvalues of $\Phi(K+1,1)$ but also its Jordan form. In the first place, it would be computationally expensive to explicitly form the matrix product $\Phi(K+1,1)$; secondly, reliable computation of the Jordan form for repeated eigenvalues is a very delicate numerical problem [4]. We comment further on this later. One of the contributions of this paper is to give a numerically attractive algorithm for solving Equations (11) and (12), which avoids the above pitfalls.

On the other hand, when $\Phi(K+1,1)$ is not invertible, i.e., when the system (6) is not reversible, the problem may have no solution. We illustrate this with two simple examples, and briefly describe what happens in each case. Both examples fail to have a floquet transform, but for different reasons.

EXAMPLE 1 [There is no Kth root $\widehat{A}$]. Take $K = 2$, and

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For this example, we need to find the square root of

$$\Phi(K+1,1) = A_2 A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

as mentioned in connection with (11). However, it can be shown easily that such a square root does not exist [3].

EXAMPLE 2 [$\widehat{A}$ exists, but $T(j)$ does not exist]. Take $K = 3$, and

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It can be verified that

$$\Phi(K+1,1) = A_3 A_2 A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this example, there are several cube roots possible, such as the zero matrix, or any matrix of the type

$$T^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} T \quad \text{or} \quad T^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} T.$$

However, it will become clear later on that, for any of these roots, there is no solution to the matrix Equations (12).

These two examples show that we have to deal with two *separate* problems when trying to find a discrete-time Floquet transform, namely those of solving (11) and (12). Conditions for the Kth roots of a matrix Φ are complicated when it is singular [3]. Moreover, the matrix Equations (12) for $T(j)$ imply row-space inclusions which seem to make our problem even worse. Yet the necessary and sufficient condition for the existence of a discrete-time Floquet transform turns out to be relatively simple, as shown in Section 3.

3. MAIN RESULT

We now state and prove the main result of this paper. It gives a necessary and sufficient condition for the system (6) to have a discrete-time Floquet transform.

THEOREM 2. *A solution to* (10) *exists iff the following rank conditions hold*:

$$\begin{aligned} \operatorname{rank}(A_{j+i-1} \cdots A_{j+1} A_j) &= r_i, \quad \textit{independent of } j, \\ & \textit{for} \quad 1 \le i \le n, \quad 1 \le j \le K. \end{aligned} \tag{13}$$

The rest of this section is devoted to a proof of this theorem. Necessity is obvious, since (9) implies

$$\widehat{A}^i = T^{-1}(j+i)A_{j+i-1}\cdots A_{j+1}A_jT(j) \qquad \forall i, j,$$

where the $T(\cdot)$ are assumed invertible. Therefore $\operatorname{rank}(A_{j+i-1}\cdots A_{j+1}A_j)$ is independent of j for all values of i and j, in particular for those mentioned in (13).

Sufficiency is proved by construction. As already indicated in Section 2, the only real difficulty in finding a discrete-time Floquet transform arises when $\Phi(K+1,1)$ is singular—this is reflected in the elaborate care needed to handle the eigenvalue at zero.

Since the proof is lengthy, we summarize here the key steps or ideas. First, we "preprocess" the system (6) via unitary transformations $Q(j)$. This preprocessing proceeds in two steps, each of which serves a distinct purpose:

1. *Deflating the eigenvalue at zero*: Not only does this step facilitate the computation of a Floquet transform, it also gives the complete Jordan structure of the zero eigenvalue of $\Phi(K+1,1)$, and implicitly of $\widehat{A}$. Hence this has important theoretical implications.
2. *Upper-triangularizing the matrices A_j*: In this step, the periodic Schur decomposition [1] is used to upper-triangularize A_j, while implicitly putting $\Phi(K+1,1)$ in Schur form. As we shall see, this greatly simplifies subsequent steps.

Then we turn to the actual construction of the Floquet transform. We find upper-triangular updating transformations $T_{\text{up}}(j)$ by solving the matrix Equations (10), which now involve only upper-triangular A_j matrices. Thus, in effect, we exhibit the overall Floquet transform for the original system (6) in the form

$$T(j) = Q(j)T_{\text{up}}(j), \qquad Q(j) \text{ unitary}, \quad T_{\text{up}}(j) \text{ upper-triangular}, \tag{14}$$

where $Q(j)$ and $T_{\text{up}}(j)$ are periodic matrices with period K. Note that this is the QR factorization of $T(j)$, a form also useful for computational purposes.

For the first step of the preprocessing, we need to consider kernels of the matrix products shown in (13), namely products of i matrices starting with A_j on the right. Define the subspaces

$$\mathcal{N}_i^{(j)} = \ker(A_{j+i-1}\cdots A_{j+1}A_j), \qquad \dim \mathcal{N}_i^{(j)} = \nu_i. \tag{15}$$

We have dropped the superscript j for ν_i because the rank condition in (13) implies that the dimension of $\mathcal{N}_i^{(j)}$ depends only on i, and not on j.

Clearly $\mathcal{N}_{i+1}^{(j)} \supset \mathcal{N}_i^{(j)}$, since $\ker(AB) \supset \ker B$ for all A, B. So ν_i is a nondecreasing sequence, $\nu_{i+1} \geq \nu_i$. Then, since $\nu_i \leq n$, there must be a smallest index k such that $\nu_i = \nu_k$ for all $i \geq k$. Define $\nu_0 = 0$, and the dimension increments

$$s_i := \nu_i - \nu_{i-1}, \qquad i = 1, \ldots, k, k+1, \ldots . \tag{16}$$

Since ν_i is nondecreasing, we have $s_i \geq 0$. Moreover, it will be shown that s_i is nonincreasing. From this, we can infer the following:

$$\nu_{k'+1} = \nu_{k'} \quad \Rightarrow \quad \nu_i = \nu_{k'} \quad \forall i \geq k'.$$

In other words, if two consecutive kernels $\mathcal{N}_{k'}^{(j)}$ and $\mathcal{N}_{k'+1}^{(j)}$ ever coincide, then all succeeding kernels are the same, and there is no further increase in dimension. This means that all s_i are in fact *strictly* greater than zero until $s_{k'+1} = 0$ for some k'. But then $k = k'$, and

$$s_1 \geq s_2 \geq \cdots \geq s_k > 0.$$

All this means is that ν_i is strictly increasing with nonincreasing increments s_i until it reaches its maximum at ν_k. Figure 2 makes this clear—it shows a typical growth of ν_i with i.

Another consequence of this is that k can never *exceed* n: even if all s_i, $i = 1, \ldots, k$, were equal to 1, no dimension increase would be possible beyond n, since $\sum_{j=1}^{i} s_j = \nu_i \leq n$. This is the reason why, in the formulation of Theorem 2, index i needs to take values 1 through n only.

We now turn to the construction of the matrices $Q(j)$. For each $j, 1 \leq j \leq K$, we choose $Q(j)$ such that

$$Q(j) = \left[\underbrace{Q_1^{(j)}}_{s_1} \middle| \underbrace{Q_2^{(j)}}_{s_2} \middle| \cdots \middle| \underbrace{Q_k^{(j)}}_{s_k} \middle| \underbrace{Q_{k+1}^{(j)}}_{n-\nu_k} \right] \tag{17}$$

is invertible, with

$$\mathcal{N}_i^{(j)} = \operatorname{span} \left[\underbrace{Q_1^{(j)} | \cdots | Q_i^{(j)}}_{\nu_i} \right], \qquad i = 1, \ldots, k.$$

This is always possible by completing bases of the growing spaces $\mathcal{N}_i^{(j)}$,

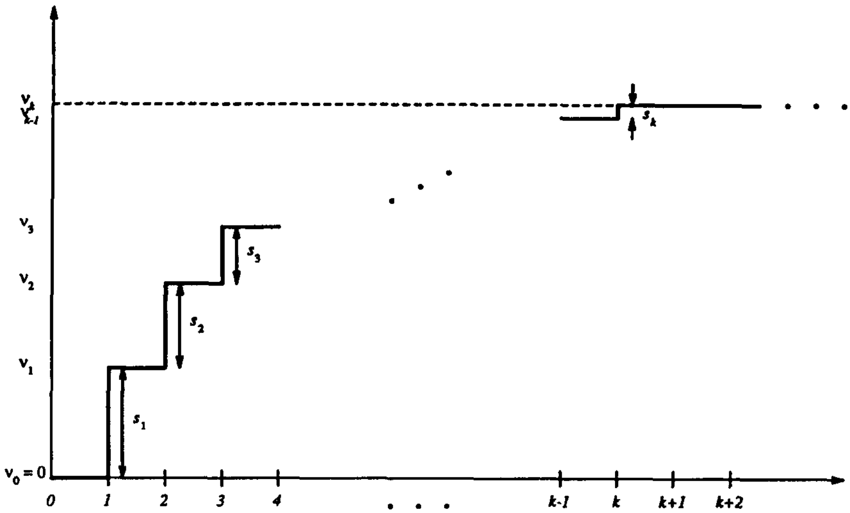

FIG. 2. Illustration of dimensions ν_i of the growing subspaces $\mathcal{N}_i^{(j)}$.

and in fact $Q(j)$ may be chosen unitary, i.e. $Q^*(j) = Q^{-1}(j)$. For efficient numerical algorithms to accomplish this, see [2].

Next we prove, by induction, that

$$Q^{-1}(j+1)A_jQ(j) = \left[\begin{array}{cccc|c} 0_{s_1} & A_{1,2}^{(j)} & \cdots & A_{1,k}^{(j)} & A_{1,k+1}^{(j)} \\ & 0_{s_2} & \ddots & \vdots & \vdots \\ & & \ddots & A_{k-1,k}^{(j)} & \vdots \\ \mathbf{0} & & & 0_{s_k} & A_{k,k+1}^{(j)} \\ \hline & & 0 & & A_{k+1,k+1}^{(j)} \end{array}\right], \qquad (18)$$

where 0_{s_i} denotes a square zero matrix of size s_i, the matrices $A_{i,i+1}^{(j)}$ have full column rank for $i = 1, \ldots, k-1$, and $A_{k+1,k+1}^{(j)}$ is invertible. Notice that the full-column-rank property tells us immediately that s_i is nonincreasing.

Firstly, from the construction of $Q(j)$, it is clear that

$$Q^{-1}(j+1)A_jQ(j) = \left[\begin{array}{c|c} 0_{s_1} & X_1 \\ \hline 0 & X_2 \end{array}\right], \qquad (19)$$

with $\left[\frac{X_1}{X_2}\right]$ of full column rank $n - s_1$. The first s_1 columns are zero because the first s_1 columns of $Q(j)$ span $\ker A_j \doteq \mathcal{N}_1^{(j)}$. Then, since $\dim \mathcal{N}_1^{(j)} = \nu_1 = s_1$, A_j has rank $n - s_1$, which is also the rank of the matrix in (19) because multiplication by invertible matrices does not change rank. In fact, the same reasoning can be applied to get the following more general result:

$$\forall i, j, \qquad Q^{-1}(j+i)A_{j+i-1}\cdots A_{j+1}A_jQ(j) = \left[\begin{array}{c|c} 0_{\nu_i} & X_1 \\ \hline 0 & X_2 \end{array}\right], \tag{20}$$

with $\left[\frac{X_1}{X_2}\right]$ of full column rank $n - \nu_i$; because the first ν_i columns of $Q(j)$ span $\mathcal{N}_i^{(j)}$, and $\mathcal{N}_i^{(j)}$ has dimension ν_i. Of course, the only interesting values for i and j in (20) are $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, K$.

For the induction step, assume that (18) holds for $i - 1$ zero diagonal blocks. That is,

$$Q^{-1}(j+1)A_jQ(j) = \left[\begin{array}{cccc|c} 0_{s_1} & A_{1,2}^{(j)} & \cdots & \cdot & A_{1,i}^{(j)} \\ & 0_{s_2} & \ddots & \vdots & \vdots \\ & & \ddots & A_{i-2,i-1}^{(j)} & \vdots \\ \mathbf{0} & & & 0_{s_{i-1}} & A_{i-1,i}^{(j)} \\ \hline & & 0 & & A_{i,i}^{(j)} \end{array}\right], \tag{21}$$

with $A_{\ell,\ell+1}^{(j)}$ full column rank for $\ell = 1, \ldots, i-2$. For the basis for induction, note that when $i - 1 = 1$, (21) reduces to (19), a form we have already established. Starting with $Q^{-1}(j+2)A_{j+1}Q(j+1)$ on the extreme right, premultiply $i - 1$ matrices each having the form shown in (21). Because of the zero-block structure in (21), this yields a product with a zero leading principal submatrix of dimension $\nu_{i-1} = \sum_{\ell=1}^{i-1} s_\ell$:

$$\begin{aligned} &Q^{-1}(j+i)A_{j+i-1}Q(j+i-1)\cdots Q^{-1}(j+2)A_{j+1}Q(j+1) \\ &= Q^{-1}(j+i)A_{j+i-1}\cdots A_{j+1}Q(j+1) = \left[\begin{array}{c|c} 0_{\nu_{i-1}} & X_1 \\ \hline 0 & X_2 \end{array}\right]. \end{aligned} \tag{22}$$

Moreover, the rank assumption (13) guarantees that $\left[\frac{X_1}{X_2}\right]$ has full column rank, since $\mathcal{N}_{i-1}^{(j+1)}$ has dimension ν_{i-1}. Note that we could also have arrived directly at (22), and the subsequent conclusion about $\left[\frac{X_1}{X_2}\right]$, by appealing to the general result (20).

Next express $Q^{-1}(j+i)A_{j+i-1}\cdots A_jQ(j)$ as a product of the matrices in (22) and (21):

$$
\begin{aligned}
&Q^{-1}(j+i)A_{j+i-1}\cdots A_{j+1}A_jQ(j)\\
&= Q^{-1}(j+i)A_{j+i-1}\cdots A_{j+1}Q(j+1)\cdot Q^{-1}(j+1)A_jQ(j)\\
&= \left[\begin{array}{c|c} 0_{\nu_{i-1}} & X_1 \\ \hline & X_2 \end{array}\right] \cdot \left[\begin{array}{cccc|c} 0_{s_1} & A^{(j)}_{1,2} & \cdots & & A^{(j)}_{1,i} \\ & \ddots & \ddots & & \vdots \\ & & & 0_{s_{i-1}} & A^{(j)}_{i-1,i} \\ \hline & & & & A^{(j)}_{i,i} \end{array}\right]\\
&= \left[\begin{array}{c|c} \underbrace{0}_{\nu_{i-1}} & \left[\frac{X_1}{X_2}\right] A^{(j)}_{i,i} \end{array}\right].
\end{aligned}
$$

Since the first $\nu_i = \nu_{i-1}+s_i$ columns of $Q(j)$ span $\mathcal{N}_i^{(j)}$, the first ν_i columns of the above matrix must be zero—as shown in (20). For this to hold, the first s_i columns of

$$\left[\frac{X_1}{X_2}\right] A^{(j)}_{i,i}$$

must be zero. But then the full-rank property of $\left[\frac{X_1}{X_2}\right]$ implies that the first s_i columns of $A^{(j)}_{i,i}$ must be zero, which is essentially (21) updated from $i-1$ to i. This induction process continues until step k, where no dimension increases are encountered anymore. This completes the proof of the claim made in (18), and ends step 1 of the preprocessing.

To recapitulate, we have at this point reduced all matrices A_j to the form shown in (18). In doing so, we isolated the zero eigenvalue of $\Phi(K+1,1) = A_K\cdots A_2A_1$ in the leading blocks of A_j, and separated off the nonsingular part in the $A^{(j)}_{k+1,k+1}$ matrices. Moreover, we obtained all this using unitary transformations only. Now we have to finish our task of constructing a Floquet transform by solving Equation (10) for $\widehat{A}$ and $T(j)$. As explained earlier, we could also set $T(1) = I$, and solve (11) for $\widehat{A}$ and (12) for $T(j)$, $j = 2,\ldots,K$. This would be strongly simplified if Equation (12) had some structure. For instance, if A_j and $\widehat{A}$ were in upper-triangular form, we could solve for upper-triangular $T(j)$ matrices. For $\widehat{A}$ to be upper-triangular, $\widehat{A}^K = \Phi(K+1,1)$ must also be so. It is relatively easy, as we shall show soon, to find the Kth root (in upper-triangular form) of an upper-triangular matrix. We need to find the eigenvalues of

$\Phi(K+1,1)$ anyway in order to compute its Kth root, and its Schur form would give us these straightaway. This leads to the question as to whether there is a numerically sound procedure which triangularizes A_j *and* puts their product $\Phi(K+1,1)$ in Schur form. It turns out that the periodic Schur decomposition [1, 5] gives precisely this result.

LEMMA 3 (Periodic Schur decomposition). *Given $n \times n$ matrices B_i, $i = 1, 2, \ldots, K$, there exist $n \times n$ unitary matrices Q_i, $i = 1, 2, \ldots, K$, such that*

$$\begin{aligned} \widetilde{B}_1 &= Q_2^* B_1 Q_1, \\ \widetilde{B}_2 &= Q_3^* B_2 Q_2, \\ &\vdots \\ \widetilde{B}_{K-1} &= Q_K^* B_{K-1} Q_{K-1}, \\ \widetilde{B}_K &= Q_1^* B_K Q_K \end{aligned}$$

are upper-triangular.

In view of the advantages mentioned above, we update the unitary matrices $Q(j)$ by post multiplying them with

$$\begin{bmatrix} I_{\nu_k} & 0 \\ 0 & Q_j \end{bmatrix},$$

given by Lemma 3 applied to the bottom matrices $A^{(j)}_{k+1,\,k+1}$ in (18) so that A_j are made upper-triangular as well.

This concludes the preprocessing part. Since we used unitary transformations only, we have not changed the numerical conditioning of the original problem. Moreover, $\Phi(K+1,1)$ is now in Schur form with the zero eigenvalue on top. This is as far as we can get using just unitary transformations. The further reduction to (the as yet undetermined) matrix $\widehat{A}$ is considered next. This requires an additional updating *upper-triangular* transformation $T_{\text{up}}(j)$ that makes all the preprocessed matrices $\widetilde{A}_j := Q^{-1}(j+1)A_jQ(j)$ equal to the same matrix $\widehat{A}$. We need to solve Equation (10), which now reads

$$T_{\text{up}}(1)\widehat{A} = \widetilde{A}_K T_{\text{up}}(K), \tag{23a}$$

$$T_{\text{up}}(j)\widehat{A} = \widetilde{A}_{j-1} T_{\text{up}}(j-1), \qquad j = 2, \ldots, K, \tag{23b}$$

for $\widehat{A}$, $T_{\text{up}}(j)$, $j = 1, 2, \ldots, K$. We choose $\widehat{A}$ to be upper-triangular, with the following form:

$$\widehat{A} = \left[\begin{array}{cccc|c} 0_{s_1} & J_2 & & & \\ & 0_{s_2} & \ddots & \text{\Large 0} & \widehat{A}_{12} \\ & & \ddots & J_k & \\ & & & 0_{s_k} & \\ \hline & & & & \widehat{A}_{22} \end{array}\right], \quad J_\ell = \left[\begin{matrix} I_{s_\ell} \\ 0 \end{matrix}\right]\Big\} s_{\ell-1}, \quad \ell = 2, \ldots, k, \tag{24}$$

where $\left[\frac{\widehat{A}_{12}}{\widehat{A}_{22}}\right]$ is still to be determined. We partition the upper-triangular matrices $T_{\mathrm{up}}(j)$ in conformity with $\widehat{A}$ as

$$T_{\mathrm{up}}(j) = \left[\begin{array}{ccc|c} T^{(j)}_{1,1} & \cdots & T^{(j)}_{1,k} & T^{(j)}_{1,k+1} \\ & \ddots & \vdots & \vdots \\ & & T^{(j)}_{k,k} & \vdots \\ \hline & & & T^{(j)}_{k+1,k+1} \end{array}\right], \tag{25}$$

where $T^{(j)}_{i,i}$ is an $s_i \times s_i$ matrix, $i = 1, 2, \ldots, k$. Equation (25) actually shows $T_{\mathrm{up}}(j)$ partitioned in two ways: a "fine partition" involving the $T^{(j)}_{p,q}$ blocks, and a "coarse partition" consisting of $k+1$ block columns. When constructing $T_{\mathrm{up}}(j)$, we will use the "fine partition" shown in (25) to compute the first k block columns only. As far as the last block column is concerned, instead of finding its $T^{(j)}_{i,k+1}$ blocks directly, we proceed from left to right constructing its columns indirectly, one at a time. This difference in approach is due to the special attention the zero eigenvalue warrants—in fact, this has been a recurring theme in our discussion.

Given the form of $\widehat{A}$ in (24), it is a simple matter to find the first k block columns of $T_{\mathrm{up}}(j)$, as shown next. Since we fixed the degrees of freedom in $\widehat{A}$ in the leading $\nu_k \times \nu_k$ block, we cannot arbitrarily choose the corresponding block of any $T_{\mathrm{up}}(j)$, and they all have to be determined. The blocks $T^{(j)}_{i,\ell}$ with $i, \ell \leq k$ are constructed in a block-row fashion as follows. Consider the relevant portion, viz. the first k entries, of the ith block row of (23), $i = k-1, k-2, \ldots, 1$:

$$
\begin{aligned}
&\left[\underbrace{0}_{\nu_i}\middle| T_{i,i}^{(j)} J_{i+1} \middle| \cdots \middle| T_{i,k-1}^{(j)} J_k \right] \\
&= \left[\underbrace{0}_{\nu_i}\middle| \widetilde{A}_{i,\,i+1}^{(j-1)} \middle| \cdots \middle| \widetilde{A}_{i,\,k}^{(j-1)} \right] \cdot \left[\begin{array}{c|ccc} X_{\nu_i} & X & \cdots & X \\ \hline & T_{i+1,\,i+1}^{(j-1)} & \cdots & T_{i+1,\,k}^{(j-1)} \\ & & \cdots & \vdots \\ & & & T_{k,\,k}^{(j-1)} \end{array}\right], \\
&\quad j = 1, \ldots, K.
\end{aligned} \tag{26}
$$

When $j = 1$, we make the (obvious) "replacement" $j - 1 \leftarrow K$ in the right-hand side of Equation (26). From this, it is clear that the matrices $T_{i,\ell}^{(j)}$ can be determined recursively, block row by block row, starting with $T_{k,\,k}^{(j)} = I_{s_k}$ chosen arbitrarily. (This is an underdetermined system of compatible equations.) When (26) is used to find the ith block row, the unknowns are all on the left-hand side and defined in terms of previously defined rows, and so can be solved for. Notice that only the first $s_{\ell+1}$ columns of each $T_{i,\ell}^{(j)}$ are defined; the rest are arbitrary. Moreover, we can freely choose the entire kth block column of $T_{\text{up}}(j)$—a convenient choice for this is $T_{k,k}^{(j)} = I$, $T_{\ell,k}^{(j)} = 0$, $1 \leq \ell \leq k$. We also have to assure that the blocks $T_{i,\,i}^{(j)}$ are invertible; but from (26) it follows that

$$T_{i,\,i}^{(j)} J_{i+1} = \widetilde{A}_{i,\,i+1}^{(j-1)} T_{i+1,\,i+1}^{(j-1)},$$

where $\widetilde{A}_{i,\,i+1}^{(j-1)}$ has full column rank. Remember that this defines (only) the first s_{i+1} columns of $T_{i,i}^{(j)}$—the remaining ones are arbitrary. If $T_{i+1,i+1}^{(j-1)}$ is invertible, these first s_{i+1} columns are linearly independent; and so we can *choose* $T_{i,i}^{(j)}$ also invertible by appropriately completing its columns.

At this stage, all matrices $\widetilde{A}_j$ are upper-triangular, and equal in their first ν_k columns. In other words, we have already applied $T_{\text{up}}(j)$ in the preliminary form

$$T_{\text{up}}(j) = T_{\text{up}}^{(1)}(j) := \left[\begin{array}{ccc|c} T_{1,1}^{(j)} & \cdots & T_{1,k}^{(j)} & \\ & \ddots & \vdots & 0 \\ & & T_{k,k}^{(j)} & \\ \hline & 0 & & I_{n-\nu_k} \end{array}\right]. \tag{27}$$

We can think of $T^{(1)}_{\mathrm{up}}(j)$ as *initializing* matrices, constructed to handle the zero-eigenvalue blocks.

We proceed further, column by column, making one additional column i (for $i = \nu_k + 1, \ldots, n$) of every $\widetilde{A}_j$ equal at each step, and updating the corresponding column of $T_{\mathrm{up}}(j)$ along the way. This amounts to postmultiplying $T^{(1)}_{\mathrm{up}}(j)$ by

$$\begin{array}{rc}
 & \begin{array}{cc} \overbrace{\qquad\qquad}^{\nu_k} & \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{\text{columns } \nu_k+1 \text{ through } n} \end{array} \\
T^{(2)}_{\mathrm{up}}(j) = \begin{array}{r} \nu_k\left\{\vphantom{\begin{matrix}1\\ \ddots\\ 1\end{matrix}}\right. \\ \vphantom{\begin{matrix}a\\a\\a\\a\\a\end{matrix}} \end{array} & \left[\begin{array}{ccc|cccc} 1 & & & & & & \\ & \ddots & & t^{(j)}_{\nu_k+1} & & & \\ & & 1 & & & & \\ & & & \tau^{(j)}_{\nu_k+1,\nu_k+1} & & t_i^{(j)} & \\ & & & & \ddots & & \\ & & & & & \tau_{ii}^{(j)} & t_n^{(j)} \\ & & & & & & \ddots \\ & & & & & & \tau_{nn}^{(j)} \end{array}\right]
\end{array}, \tag{28}$$

where $\tau_{ii}^{(j)}$ are nonzero scalars, $t_i^{(j)}$ are $(i-1)$-vectors, and the columns $\left[\begin{smallmatrix} t_i^{(j)} \\ \tau_{ii}^{(j)} \end{smallmatrix}\right]$ are determined one at a time. Now there is no knowledge of $\widehat{A}$ to guide us; therefore we must solve for its columns $\nu_k + 1$ through n as well. Since we do not need to put $\widehat{A}$ in special form anymore, it is easier to fix $T^{(2)}_{\mathrm{up}}(1) \equiv I_n$, and hence determine the ith column of $\widehat{A}$ from the equation

$$\widehat{A}^K = \widetilde{A}_K \cdots \widetilde{A}_2 \widetilde{A}_1. \tag{29}$$

Notice that since $T^{(2)}_{\mathrm{up}}(1) \equiv I_n$, this product does not change anymore in the sequel.

Let us denote $S := \widetilde{A}_K \cdots \widetilde{A}_2 \widetilde{A}_1$, and construct first the ith column of $\widehat{A}$. For this, it is helpful to partition the leading principal submatrices of order i of $\widehat{A}$ and S as follows:

$$\widehat{A}_{[i]} = \left[\begin{array}{c|c} \widehat{A}_{[i-1]} & \widehat{a}_i \\ \hline 0 & \widehat{\alpha}_{ii} \end{array}\right], \qquad S_{[i]} = \left[\begin{array}{c|c} S_{[i-1]} & s_i \\ \hline 0 & \sigma_{ii} \end{array}\right], \tag{30}$$

where $\widehat{a}_i, s_i$ are $(i-1)$-vectors, $\widehat{\alpha}_{ii}, \sigma_{ii}$ are scalars, and $\widehat{A}_{[i-1]}, S_{[i-1]}$ are upper-triangular matrices of size $i-1$. Of course, $\widehat{A}_{[i-1]}$ and $S_{[i-1]}$ are also the leading principal submatrices of order $i-1$ of $\widehat{A}$ and S respectively. The diagonal element σ_{ii} is nonzero, since it is the product of the ith diagonal elements of the matrices $\widetilde{A}_j$:

$$\sigma_{ii} = \prod_{j=1}^{K} \widetilde{\alpha}_{ii}^{(j)},$$

where $\widetilde{\alpha}_{ii}^{(j)} \neq 0$ is the ith diagonal element of $\widetilde{A}_j$. We see that (29) implies

$$\begin{aligned} \widehat{A}_{[i-1]}^{K} &= S_{[i-1]}, \\ \widehat{\alpha}_{ii}^{K} &= \sigma_{ii}, \end{aligned} \tag{31}$$

and

$$\left(\widehat{A}_{[i-1]}^{K-1} + \widehat{A}_{[i-1]}^{K-2}\widehat{\alpha}_{ii} + \cdots + \widehat{\alpha}_{ii}^{K-1} I\right)\widehat{a}_i = s_i. \tag{32}$$

Since $\widehat{A}_{[i-1]}$ is known when solving for the ith column of $\widehat{A}$, and $\widehat{\alpha}_{ii}$ is given by (31), we can solve for $\widehat{a}_i$ from Equation (32). Note that $\widehat{\alpha}_{ii}$ is nonzero, since σ_{ii} is nonzero. Negative σ_{ii} is not a problem because we are using complex arithmetic. Therefore $\widehat{A}_{22}$ in (24) is an invertible matrix–this is an important observation, and will be useful when solving (23b). The coefficient matrix of (32) is upper-triangular, with its mth diagonal entry given by

$$d_m = \sum_{\ell=0}^{K-1} \widehat{\alpha}_{mm}^{\ell} \widehat{\alpha}_{ii}^{K-1-\ell} = \frac{\widehat{\alpha}_{mm}^{K} - \widehat{\alpha}_{ii}^{K}}{\widehat{\alpha}_{mm} - \widehat{\alpha}_{ii}}, \quad m = 1, 2, \ldots, i-1.$$

Clearly, $d_m = 0$ iff $(\widehat{\alpha}_{mm}^{K} = \widehat{\alpha}_{ii}^{K}$ and $\widehat{\alpha}_{mm} \neq \widehat{\alpha}_{ii})$. Thus (32) can be solved uniquely for $\widehat{a}_i$ if S has distinct eigenvalues, or if we take $\widehat{\alpha}_{ii} = \widehat{\alpha}_{mm}$ whenever $\sigma_{ii} = \sigma_{mm}$.

Once the ith column of $\widehat{A}$ is determined, we proceed with the corresponding column of $T_{\text{up}}^{(2)}(j)$. As mentioned earlier, assume that all the initializing transformations $T_{\text{up}}^{1}(j)$ (27), have been applied to the matrices $\widetilde{A}_j$ already; and determine $T_{\text{up}}^{(2)}(j)$, $j = 2, \ldots, K$, using (23b). We reproduce (23b) here, suitably modified:

$$T_{\text{up}}^{(2)}(j)\widehat{A} = \widetilde{A}_{j-1} T_{\text{up}}^{(2)}(j-1), \qquad j = 2, \ldots, K,$$

and rewrite it with the leading principal submatrices partitioned as in (30):

$$\left[\begin{array}{cc|c} T^{(2)(j)}_{\mathrm{up},[i-1]} & t_i^{(j)} & \mathrm{x} \\ 0 & \tau_{ii}^{(j)} & \mathrm{x} \\ \hline & & \mathrm{X} \end{array}\right] \cdot \left[\begin{array}{cc|c} \widehat{A}_{[i-1]} & \widehat{a}_i & \mathrm{x} \\ 0 & \widehat{\alpha}_{ii} & \mathrm{x} \\ \hline & & \mathrm{X} \end{array}\right]$$
$$= \left[\begin{array}{cc|c} \widetilde{A}^{(j-1)}_{[i-1]} & \widetilde{a}_i^{(j-1)} & \mathrm{x} \\ 0 & \widetilde{\alpha}_{ii}^{(j-1)} & \mathrm{x} \\ \hline & & \mathrm{X} \end{array}\right] \cdot \left[\begin{array}{cc|c} T^{(2)(j-1)}_{\mathrm{up},\,[i-1]} & t_i^{(j-1)} & \mathrm{x} \\ 0 & \tau_{ii}^{(j-1)} & \mathrm{x} \\ \hline & & \mathrm{X} \end{array}\right], \; j = 2, \ldots, K.$$

For the diagonal elements, this yields

$$\tau_{ii}^{(j)} \widehat{\alpha}_{ii} = \widetilde{\alpha}_{ii}^{(j-1)} \tau_{ii}^{(j-1)}, \qquad j = 2, \ldots, K, \tag{33}$$

where $\widehat{\alpha}_{ii}$ and $\widetilde{\alpha}_{ii}^{(j-1)}$ are the ith diagonal elements of $\widehat{A}$ and $\widetilde{A}_{j-1}$ respectively. Since $\widehat{\alpha}_{ii}$ and $\widetilde{\alpha}_{ii}^{(j-1)}$ are nonzero, and the first diagonal element $\tau_{ii}^{(1)} = 1$, all $\tau_{ii}^{(j)}$ will be nonzero. We find the column vectors $t_i^{(j)}$ from

$$T^{(2)(j)}_{\mathrm{up},[i-1]} \widehat{a}_i + \widehat{\alpha}_{ii} t_i^{(j)} = \widetilde{A}^{(j-1)}_{[i-1]} t_i^{(j-1)} + \tau_{ii}^{(j-1)} \widetilde{a}_i^{(j-1)}, \qquad j = 2, \ldots, K, \tag{34}$$

which is again solvable since $\widehat{\alpha}_{ii}$ is nonzero. Since we thereby constructed the transformations $T_{\mathrm{up}}(j)$, and they are invertible by construction, this concludes the proof of Theorem 2.

REMARK 4. We could have required $\widehat{A}_{22}$, the invertible part of $\widehat{A}$, to be in Jordan canonical form as well, but this would lead to a more complicated construction of the matrices $T_{\mathrm{up}}(j)$. In the case of the zero eigenvalue, this was actually needed, since existence conditions depended on it. Of course, after $\widehat{A}$ has been determined, we can still continue to reduce it to Jordan form using the methods described in [8] or [4].

REMARK 5. Going back to the two examples given in Section 2, it is clear that the first one violates the rank condition (13) for $i = 1$ itself, since $\operatorname{rank} A_1 \neq \operatorname{rank} A_2$. The second example, on the other hand, violates (13) only for $i = 3$. Note that rank A_1 = rank A_2 = rank $A_3 = 2$ and rank $A_2 A_1$ = rank $A_3 A_2$ = rank $A_1 A_3 = 1$; but on taking three-matrix products, we get

$$A_2A_1A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1A_3A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_3A_2A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and it is obvious that these have different ranks. Hence as claimed before, neither example admits a discrete-time Floquet transform.

4. CONCLUDING REMARKS

We have given necessary and sufficient conditions for the existence of a Floquet transform for a linear homogeneous periodic difference equation or discrete-time state-space system (6). The implications of this result are important. If a Floquet transform exists, then the behavior of the periodic system is essentially described by that of a time-invariant one. Our result shows that this equivalence actually does not always exist, in contrast to the continuous-time case (where it is well known to exist).

Similar results can also be obtained for implicit difference equations (or generalized state-space systems) of the type

$$E_j x(j+1) = A_j x(j), \qquad x(0) = x_0 \tag{35}$$

with $E_j = E_{j+K}, A_j = A_{j+K}\ \forall j$ (K minimal). In this case, the existence of a Floquet transform will require rank conditions to hold that describe the behavior of both the zero and infinite (generalized) eigenvalue.

These results are expected to lead to precise conditions for the solution of certain discrete-time control problems of periodic systems. Indeed, when a Floquet transform of such a system exists, it is easy to see that the problem can essentially be reduced to one involving a time-invariant state equation, for which a number of results are known.

The results of this paper also implicitly give a numerical method for computing a Kth root of a matrix product, and show the importance of the periodic Schur form in this context. Again in the context of generalizing this to the system (35), the periodic Schur form will play a similar role.

Finally, we note that in the case of real arithmetic, the problem is slightly different. The periodic Schur form can be found using real arithmetic only, but one of the matrices A_j as well as the monodromy matrix

Φ will have 2×2 diagonal blocks (one for each pair of complex eigenvalues). The problem of finding a Kth real root for those 2×2 blocks can still be solved, but if Φ has a negative (and hence real) eigenvalue of *odd* multiplicity—even without Jordan blocks—then complex eigenvalues will appear in the Kth root $\widehat{A}$ which cannot all be paired with complex conjugate ones into a real matrix. When Φ has Jordan blocks of a negative (real) eigenvalue, the same reasoning applies: the number of Jordan chains of equal length should be *even*, in order to find real roots. Of course, in such cases, we can always double the period K (thereby squaring Φ): this will take care of the problem, since then a real Kth root $\widehat{A}$ exists. This is akin to the continuous-time situation, where one can always choose R in Equation (2) to be real by considering the state transition matrix over *twice* the basic period T—then the state transition matrix over one (new) period $T' = 2T$ is expressible as a matrix squared,

$$\Phi(T', 0) = \Phi(T, 0)\Phi(T, 0),$$

and therefore admits a real logarithm [7, 9]. Note that then the Floquet transform $P(t)$ is also real, but with period $T' = 2T$.

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REFERENCES

1 A. Bojanczyk, G. Golub, and P. Van Dooren, The periodic Schur decomposition. Algorithms and applications, in *Proceedings of the SPIE Conference* San Diego, Vol. 1770, July 1992, pp. 31–42.

2 B. De Moor and P. Van Dooren, Generalizations of the singular value and QR decompositions, *SIAM J. Matrix Anal. Appl.* 13:993–1014 (Oct. 1992).

3 F. R. Gantmacher, *The Theory of Matrices,* Vol. 1, Chelsea, New York, 1959.

4 G. H. Golub and J. H. Wilkinson, Ill-conditioned eigensystems and the computation of the Jordan canonical form, *SIAM Rev.* 18:579–619 (1976).

5 J. J. Hench, *Numerical Methods for Periodic Linear Systems,* Ph.D. Thesis, Univ. of California, Santa Barbara, Sept. 1992.

6 J. J. Hench and A. J. Laub, An Extension of Floquet Theory to Discrete-Time Periodic Systems, Technical Report CCEC-92-0914, Univ. of California, Santa Barbara, 1992.
7 V. A. Ĭakubovich and V. M. Starzhinskiĭ, *Linear Differential Equations with Periodic Coefficients*, Vol. 1, Wiley, Halsted Press, New York, 1975.
8 B. Kågström and A. Ruhe, An algorithm for numerical computation of the Jordan normal form of a complex matrix, *ACM Trans. Math. Software* 6:398–419 (1980).
9 L. S. Pontryagin, *A Course in Ordinary Differential Equations,* Internat. Monogr. Adv. Math. & Phys., Hindustan Publishing Corp., Delhi, 1967.
10 J. Sreedhar and P. Van Dooren, Discrete-time periodic systems: A Floquet approach, in *Proceedings of the Conference on Information Sciences & Systems,* Baltimore, 24–26 Mar. 1993, pp. 194–195.

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