According to Definition 1, a straightforward calculation shows that for $i = 1, \cdots, l,\ j = 1, \cdots, m$

$$\begin{aligned} L_{\overline{f}}^{s} V_i^0(x_1, t) &= V_i^s(x_1, t), \qquad s = 0, \cdots, \rho_i - 1 \\ L_{\overline{f}}^{s} V_i^0(x_1, t) &= 0, \qquad s = \rho_i, \rho_i + 1, \cdots \qquad (34) \\ dV_i^s(x_1, t)\overline{g}_j(x_1, t) &= 0, \qquad s = 0, \cdots, \rho_i - 1. \qquad (35) \end{aligned}$$

Moreover, it follows from (33) that for $i = 1, \cdots, h,\ j = 1, \cdots, m$

$$L_{\overline{f}}^{s} U_i^0(x_1, t) = U_i^s(x_1, t), \qquad s = 0, \cdots, r_i - 1 \qquad (36)$$

$$dV_i^s(x_1, t)\overline{g}_j(x_1, t) = \begin{cases} 0, & s \le r_i - 2 \\ \hat{c}_{ij}(t), & s = r_i - 1 \end{cases} \qquad (37)$$

with $\hat{c}_{ij}(t)$ denoting the jth column of $\hat{c}_i(t)$.

According to [11, Lemma 4.1.2], it follows from (34), (35) that

$$\left[M_i^s(t) \quad \frac{dM_i^s(t)}{dt} x_1 \right] ad_{\overline{f}}^{k} \overline{g}_j(x_1, t) = 0, \quad s = 0, \cdots, \rho_i - 1, k = 1, \cdots, \hat{r} \qquad (38)$$

for $i = 1, \cdots, l,\ j = 1, \cdots, m$, which means $Q\tilde{P} = 0$, and it follows from (36), (37) that

$$\left[T_i^s(t) \quad \frac{dT_i^s(t)}{dt} x_1 \right] ad_{\overline{f}}^{k} \hat{g}_j(x_1, t) = \begin{cases} 0, & s + k \le r_i - 2 \\ (-1)^{r_i - s - 1}\hat{c}_{ij}(t), & s + k = r_i - 1 \end{cases}$$

for $i = 1, \cdots h,\ j = 1, \cdots, m$, which implies that $\tilde{Q}\tilde{P}$ admits a block triangular structure after reordering its rows and its diagonal blocks consist of rows of $\hat{c}(t)$. So it follows from the invertibility of $\hat{c}(t)$ that $\tilde{Q}\tilde{P}$ has full row rank. Thus for any x_1 and any $t \in [0, T]$, $\overline{QP} = \begin{bmatrix} 0 & QP \\ \tilde{Q}\tilde{P} & \tilde{Q}P \end{bmatrix}$ has full row rank, which implies that the rows of $\overline{Q}$ are linearly independent. As a consequence, the vectors in Lemma 2 are linearly independent for any $t \in [0, T]$.

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On Sampling Without Loss of Observability/Controllability

Gerhard Kreisselmeier

***Abstract*—This paper presents a (nonequidistant, periodic) sampling pattern, which has the property that the sample and zero-order hold operations, applied to any observable and controllable continuous-time system of order not exceeding N, results in a discrete-time system which is also observable and controllable.**

***Index Terms*—Discrete-time systems, lifting techniques, periodic sampling, sampled data systems, sampling time.**

I. INTRODUCTION

When a continuous-time system is converted into discrete-time form by means of a sample and hold operation, then observability and/or controllability can get lost [1]. If such a loss involves the unstable subsystem then the discretized system is not output feedback stabilizable. Therefore, it has always been of interest that sampling, or discretization to be more precise, is done in such a way that observability and controllability carry over from the continuous-time to the discrete-time system.

Discretization with an equidistant time pattern and a zero-order hold is almost exclusively used in practice and in the literature and is fairly completely understood mathematically (see, e.g., [1] and [2]).

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The author is with the University of Kassel, Department of Electrical Engineering, D-34109 Kassel, Germany.

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A condition on the sampling time T, which is sufficient (and in many cases also necessary) to rule out pathological situations, is that

$$(\lambda_p - \lambda_q)T \neq (2\pi j)\ell, \qquad \ell = \pm 1, \pm 2, \cdots$$

for each pair (λ_p, λ_q) of eigenvalues of the continuous-time system.

The above condition imposes only a minor restriction on the admissible sampling times when the system data are known exactly. But when the system data (and hence the relevant eigenvalues) are uncertain and only known to lie within some bounded set, as, e.g., in robust control problems, then a sufficiently small T or, equivalently, a sufficiently high sampling rate $1/T$ may have to be chosen to satisfy this condition. And when the set of uncertainty is unbounded, as, e.g., in adaptive control problem formulations, then a feasible T may not at all exist.

This background strongly motivates our interest in a sampling or discretization scheme which has the property that for each observable and controllable system, no matter what its parameters are, the resulting discrete-time system is also observable and controllable. The above discussion indicates that a solution, if one exists, must involve nonequidistant sampling. To the author's knowledge suitable nonequidistant time patterns, which rule out pathological discretization completely, have not been reported in the literature.

For completeness we note that so-called generalized hold functions can be designed for any sampling time T, so that controllability is preserved for a given system [2], [3] and, as an extension thereof, for almost any T so that controllability is preserved simultaneously for almost any given (finite) set of systems [4]. Apparently these results do not extend to the (infinite) set of all controllable systems, which is of interest here, and they also do not extend to the observability problem, due to lack of an output element, which is dual to the hold function. On the other hand, multi-rate sampling approaches (see, e.g., [5] for a comprehensive treatment and further references), which can approximate generalized hold functions, and which have the duality that multiple rates can be used both at the input and output, use equidistant high-rate sampling. So in both approaches pathological discretization is not structurally ruled out, which takes us back to the original problem.

The problem is best illustrated by an example. The signal $y(t) = \sin(\omega t)$ is the output of a second-order observable system with parameter ω and eigenvalues $\pm j\omega$. Suppose that $y(t)$ is sampled at time instants $kT, k = 0, 1, 2, \cdots$, and that the parameter value happens to be $\omega = \pi/T$. Then $y(kT) = 0$ for all k, the discretized system is not observable, and the eigenvalue condition is violated. It is clear that some extra, intermediate sampling times should be added. It is also clear that for a fixed pair of eigenvalues the eigenvalue condition cannot be violated for both T and T', where T and T' are sampling times such that T/T' is an irrational real number. This motivates sampling at times kT *and* $kT', k = 0, 1, 2, \cdots$. It is evident that this sampling pattern resolves the observability problem in the above example for all ω. The underlying idea, which is to observe those eigenmotions at times kT' which cannot be observed at times kT, and vice versa, however, does not carry over to higher order systems. For example, the signal

$$y(t) = \sin(t) - \sin((1 + 2\pi)t) - \sin((1 + \sqrt{2}\pi)t) + \sin((1 + \sqrt{2}\pi + 2\pi)t)$$

is the output of an eighth-order observable system. Sampling it at times kT and kT' with $T = 1, T' = \sqrt{2}$ gives zero at all sampling instants, and so observability is not preserved. This illustrates some of the complexity and counterintuitiveness of the problem.

In this paper we go back to the original definition of observability and derive a simple nonequidistant periodic sampling pattern. It has the most notable property that new measurement samples always contain new information so that, in the long run, the accumulated information is the same as if the output had been sampled *densely* in time. This clearly preserves observability. The case of controllability is then the dual. Although different in equations, it follows the same principle.

II. Main Result

Consider a linear, continuous time system

$$\Sigma_C: \quad \dot{x} = Fx + Gu, x(0) = x_0$$
$$y = Hx \tag{1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^\ell$, and time $t \in \mathbb{R}^+$.

Consider further a sampling pattern $\{t_k, k \in \mathbb{Z}_0^+\}$ and a zero-order hold applied to (1), which results in the associated discretized system

$$\Sigma_D: x_{k+1} = e^{F(t_{k+1}-t_k)}x_k + \int_{t_k}^{t_{k+1}} e^{F(t_{k+1}-\tau)}G\, d\tau\; u_k$$
$$y_k = Hx_k \tag{2}$$

where $x_k := x(t_k)$ and $y_k := y(t_k)$, and u_k denotes the (constant) input for $t \in [t_k, t_{k+1})$.

In order that observability and controllability can always (i.e., irrespective of the actual system data) carry over from Σ_C to Σ_D, the sampling pattern will have to be nonequidistant, and therefore Σ_D will be a time varying system. Its observability and controllability shall be established in the following sense.

Definition (Observability): Σ_D is said to be (uniformly and completely) observable, if there exists a positive integer ν such that, given any initial time t_k, the state x_k is uniquely determined from $\{y_k, y_{k+1}, \cdots, y_{k+\nu}\}$ and $\{u_k, u_{k+1}, \cdots, u_{k+\nu-1}\}$. □

Definition (Controllability): Σ_D is said to be (uniformly and completely) controllable, if there exists a positive integer ν such that, given any initial time t_k, initial state x' and terminal state x'',, there is a control sequence $\{u_k, u_{k+1}, \cdots, u_{k+\nu-1}\}$ which transfers the state from $x_k = x'$ to $x_{k+\nu} = x''$. □

The property of completeness refers to the fact that ν is independent of the initial and terminal states, while uniformity requires that the same ν works for all initial times. Both properties together make observability and controllability of Σ_D qualitatively similar to that of a time-invariant system [6], [7].

Finally we propose a sampling pattern $\{t_k, k \in \mathbb{Z}_0^+\}$ with structure

$$t_k = \begin{cases} kT, & k = 0, 1, 2, \cdots, N \\ NT + T', & k = N + 1 \\ t_{k-(N+1)} + NT + T', & k > N + 1 \end{cases} \tag{3}$$

and any choice of parameters $T, T' \in \mathbb{R}^+$ and $N \in \mathbb{Z}^+$ such that

$$T/T' \text{ is not rational} \tag{4}$$

and

$$N \geq n. \tag{5}$$

As usual $\mathbb{R}^+, \mathbb{Z}, \mathbb{Z}^+$ and $\mathbb{Z}_0^+$ denote the sets of positive real numbers, integers, positive integers, and nonnegative integers, respectively.

The sampling pattern is illustrated in Fig. 1. It is nonequidistant and periodic. Each period has a block of N successive sampling intervals of length T, followed by one sampling interval of length T'.

Based on these preliminaries the main result of this paper can now be stated as follows.

Theorem: If Σ_C is discretized with sampling pattern (3)–(5), then:

1) observability of Σ_C implies observability of Σ_D;
2) controllability of Σ_C implies controllability of Σ_D. □

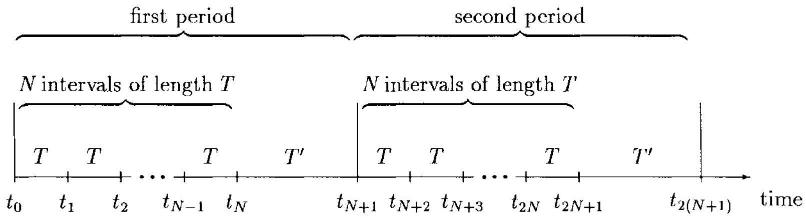

Fig. 1. The proposed nonequidistant periodic sampling pattern.

For completeness we note that if Σ_C is not observable or not controllable, then the Theorem still applies to the observable subsystem and to the controllable subsystem of Σ_C, respectively. The result can then be summarized that in the transition from Σ_C to Σ_D, a loss of observability or controllability does not occur.

The proof of the Theorem, which is given in the next section, simplifies by the fact that the sampling pattern is periodic. This makes Σ_D periodically time-varying and hence all its properties uniform in time. So what remains to be proved is the existence of an integer ν in the sense of the above definition, where now instead of any t_k only the initial time $t_0 = 0$ needs to be considered.

III. Proof of the Theorem

A. Observability

We start by realizing that Σ_D is not observable if and only if the set of equations

$$y_k = He^{Ft_k}x_0, \qquad k \in \mathbb{Z}_0^+ \tag{6}$$

does not have a unique solution x_0 or, equivalently, if and only if there exists a nonzero vector $\xi \in \mathbb{R}^n$ such that

$$He^{Ft_k}\xi = 0, \quad \text{for all } k \in \mathbb{Z}_0^+. \tag{7}$$

The sampling pattern under investigation has blocks of N successive sampling intervals of length T. This gives rise to the following extension principle.

Lemma 1: For each $t' \in \mathbb{R}$, if

$$He^{F(t'+iT)}\xi = 0 \tag{8}$$

is true for $i = 0, 1, 2, \cdots, n-1$, then it is true for all $i \in \mathbb{Z}$. □

A proof of Lemma 1 is given in the Appendix.

Lemma 1 reflects the known fact that when output measurements of an unforced linear continuous-time system are taken equally spaced in time, then n successive measurements contain as much information (about the initial state) as infinitely many of them.

The sampling pattern further has an extra sampling interval T' at the end of each period, which causes a nonrational shift between every two successive blocks of N sampling intervals of length T. The effect is that if we let t' in Lemma 1 range over the set of all starting points of a sampling period, i.e., let $t' = j(NT + T'), j \in \mathbb{Z}_0^+$, then $t' + iT, i \in \mathbb{Z}$ ranges over all time instances $iT + jT', i \in \mathbb{Z}, j \in \mathbb{Z}_0^+$. The latter cover the time interval $[0, \infty)$ densely, as the following lemma shows.

Lemma 2: For each $t \geq 0$ and each $\epsilon > 0$ there exist $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_0^+$ such that

$$|t - (iT + jT')| < \epsilon. \tag{9}$$

□

A proof of Lemma 2 is given in the Appendix.

Based on these results, the first part of the Theorem can now be proved as follows.

Proof of the Theorem (Part 1), Observability): Suppose that Σ_D is not observable and (7) holds for some nonzero $\xi \in \mathbb{R}^n$. From (7) and the definition of the sampling times t_k in (3)–(5), it is clear that Lemma 1 applies for all starting points of a sampling period, i.e., for all $t' = j(NT + T'), j \in \mathbb{Z}_0^+$. This gives

$$He^{F(iT+jT')}\xi = 0, \quad \text{for all } i \in \mathbb{Z}, j \in \mathbb{Z}_0^+. \tag{10}$$

Since the matrix exponential is an analytic function, we can further apply Lemma 2 to conclude that

$$He^{Ft}\xi = 0, \quad \text{for all } t \geq 0 \tag{11}$$

i.e., Σ_C is not observable.

We have thus shown that if Σ_D is not observable then Σ_C is not observable. Hence observability of Σ_C implies observability of Σ_D. □

B. Controllability

The state of Σ_D at time $t_\nu, \nu \in \mathbb{Z}_0^+$ is obtained from (2) as

$$x_\nu = e^{F(t_\nu - t_0)}x_0 + \sum_{k=0}^{\nu-1}\int_{t_k}^{t_{k+1}} e^{F(t_\nu - \tau)}G\, d\tau\ u_k. \tag{12}$$

This can be rewritten in the form

$$e^{-Ft_\nu}x_\nu - e^{-Ft_0}x_0 = \sum_{k=0}^{\nu-1}\int_{t_k}^{t_{k+1}} e^{-F\tau}G\, d\tau\ u_k. \tag{13}$$

For the controllability investigation, x_ν and x_0 are allowed to be arbitrary, and so the left-hand side of (13) can assume any vector in $\mathbb{R}^n$.

The right-hand side of (13) admits (by variation of the control sequence $\{u_0, u_1, \cdots, u_{\nu-1}\}$) a certain set of vectors, which depends on ν. This set is monotone increasing with ν and for each ν forms a subspace of $\mathbb{R}^n$. If for some finite ν the corresponding subspace equals $\mathbb{R}^n$, then Σ_D is clearly controllable.

Conversely, if Σ_D is not controllable, then the right-hand side of (13) is in some fixed subspace for all $\nu \in \mathbb{Z}_0^+$, i.e., a nonzero $\xi \in \mathbb{R}^n$ exists such that

$$\int_{t_k}^{t_{k+1}} \xi^T e^{-F\tau}G\, d\tau = 0, \quad \text{for all } k \in \mathbb{Z}_0^+ \tag{14}$$

or, equivalently

$$\int_0^{t_k} \xi^T e^{-F\tau}G\, d\tau = 0, \quad \text{for all } k \in \mathbb{Z}_0^+. \tag{15}$$

The particular sampling pattern (3)–(5) now gives rise to an extension principle in integral form, which may be regarded as a controllability counterpart of Lemma 1.

Lemma 3: For each $t' \in \mathbb{R}$, if

$$\int_0^{t'+iT} \xi^T e^{-F\tau}G\, d\tau = 0 \tag{16}$$

is true for $i = 0, 1, 2, \cdots, n$, then it is true for all $i \in \mathbb{Z}$. □

A proof of Lemma 3 is given in the Appendix.

Based on this result the second part of the Theorem can now be proved as follows.

Proof of the Theorem (Part 2), Controllability): Suppose that Σ_D is not controllable. Then for some nonzero $\xi \in \mathbb{R}^n$

$$\int_0^{t_k} \xi^T e^{-F\tau} G \, d\tau = 0, \quad \text{for all } k \in \mathbb{Z}_0^+. \tag{17}$$

Writing $t_k = j(NT + T') + iT$, where $j \in \mathbb{Z}_0^+$ and $i \in \{0, 1, 2, \cdots, N\}$, we can apply Lemma 3 with $t' = j(NT + T')$ to obtain

$$\int_0^{iT+jT'} \xi^T e^{-F\tau} G \, d\tau = 0, \quad \text{for all } j \in \mathbb{Z}_0^+, i \in \mathbb{Z}. \tag{18}$$

Since $iT + jT'$ for $j \in \mathbb{Z}_0^+, i \in \mathbb{Z}$ covers the positive real line densely by Lemma 2, it follows that

$$\int_0^{t} \xi^T e^{-F\tau} G \, d\tau = 0, \quad \text{for all } t \in \mathbb{R}^+. \tag{19}$$

This implies that $\xi^T e^{Ft} G \equiv 0$ and hence that Σ_C is not controllable.

As a result, controllability of Σ_C implies controllability of Σ_D. □

IV. The Number of Steps to Observe/Control

For possible applications of the above result, the number of steps ν, which allows (uniform and complete) observation or control, is of major significance. The following corollary specifies one such ν.

Corollary: If Σ_C is observable (controllable) and the sampling pattern (3)–(5) is used, then Σ_D is observable (controllable) with $\nu = (N+1)n$. □

A proof of the Corollary is given in the Appendix.

The Corollary implies that ν is well bounded and that *all* observable (controllable) systems of order not exceeding N can be observed (controlled) in $(N+1)N$ steps. Moreover, if Σ_D is rewritten as a multirate system over one period, then the resulting time-invariant system will be observable (controllable).

V. Rational Sampling Time Ratio T/T'

Consider a sampling pattern with structure (3) and parameters T/T' *rational* and $N \geq n$. In order to analyze this case, we need the following rational counterpart of Lemma 2.

Lemma $2'$: Let $T/T' = p/q$, where $p, q \in \mathbb{Z}^+$ are coprime, and define $\Delta T := T'/q (= T/p)$. Then

$$\{iT + jT' | i \in \mathbb{Z}, j \in \mathbb{Z}_0^+\} = \{k \cdot \Delta T | k \in \mathbb{Z}\}.$$

□

A proof of Lemma $2'$ is given in the Appendix.

The sampling result for the rational case can now be stated in the subsequent Theorem. Its proof follows from Lemmas 1, $2'$, and 3 in essentially the same way as the proofs in Section III and is therefore omitted.

Theorem: If Σ_C is discretized using a sampling pattern with structure (3) and parameters T/T' *rational* and $N \geq n$, then Σ_D is:

1) observable if and only if $(H, e^{F\Delta T})$ is an observable pair;
2) controllable if and only if $(e^{F\Delta T}, \int_0^{\Delta T} e^{F(\Delta T - \tau)} G \, d\tau)$ is a controllable pair. □

The Theorem says that the proposed sampling scheme with a rational T/T' has an effect which is equivalent to that of equidistant sampling with the fictitious sampling time ΔT. Note that (while T and T' can be kept roughly the same) ΔT can be made small by taking p and q large, and the nonrational case results in the limit as $(p, q) \to \infty$, respectively, $\Delta T \to 0$.

VI. Conclusions

A sampling scheme is presented, which guarantees that sampling is *always* without loss of observability and controllability. With this scheme the (average) sampling time is now *completely arbitrary*, whatever the system parameters are.

The significance of the theoretical result is that it holds for the set of *all* nth-order observable (controllable) systems, a set which is unbounded and open, and this appears to be why the nonrational sampling time ratio T/T' enters the picture.

In practice Σ_C is typically from a compact subset, in which case (since the mapping $(\Sigma_C, T, T') \to \Sigma_D$ is continuous) the result is maintained under sufficiently small perturbations of T and T'. As a consequence, irrational ratios need not be implemented exactly in practice. Moreover, for *rational* sampling time ratios T/T', the new sampling scheme has an easy interpretation in terms of classical equidistant sampling with a (fictitious) sampling time ΔT, the size of which can be controlled by T/T'.

The sampling pattern is nonequidistant and periodic, and therefore the discretized system is periodically time-varying. Feedback laws can be designed either directly for the time-varying system or by first condensing the equations of one period into one time-invariant equation, as in lifting or multirate sampling techniques and then making a time-invariant design.

A possible impact of this result on the choice of sampling patterns in practice, on sampling rate selections, on robustness questions, or on how it can possibly be used in an adaptive control context, remain interesting topics for further research.

Appendix

Proof of Lemma 1: Using the notations $A := e^{FT}$ and $\xi' = e^{Ft'}\xi$, the assumption can be written as

$$HA^i\xi' = 0, \quad \text{for } i = 0, 1, 2, \cdots, n-1. \tag{20}$$

From the Cayley–Hamilton Theorem and the fact that A is a matrix exponential which is nonsingular, it follows that for each $i \in \mathbb{Z}$, A^i is a linear combination of $A^0, A^1, \cdots, A^{n-1}$. This can be used together with (20) to conclude that $HA^i\xi' = 0$ for all $i \in \mathbb{Z}$, which proves the Lemma. □

Proof of Lemma 2: For each $r \in \mathbb{Z}_0^+$ we can write rT' as a (nonnegative) integer multiple of T plus a remainder less than T, i.e.,

$$rT' = s_r T + \Delta_r; s_r \in \mathbb{Z}_0^+, \Delta_r \in [0, T). \tag{21}$$

The sequence $\{\Delta_r, r \in \mathbb{Z}_0^+\}$ is bounded, and therefore it has an accumulation point. Consequently, for each $\epsilon > 0$ there exist $r_1, r_2 \in \mathbb{Z}_0^+$ such that $r_1 < r_2$ and $|\Delta_{r_1} - \Delta_{r_2}| < \epsilon$. So we have

$$\begin{aligned}\delta &:= \Delta_{r_2} - \Delta_{r_1} \\ &= (r_2 - r_1)T' - (s_{r_2} - s_{r_1})T \in (-\epsilon, \epsilon)\end{aligned} \tag{22}$$

where $r_2 - r_1$ and $s_{r_2} - s_{r_1}$ are both in $\mathbb{Z}_0^+$, and $\delta \neq 0$ because T'/T is not rational.

If $\delta > 0$, then we can write t as a (nonnegative) integer multiple of δ plus a remainder less than ϵ, which gives the desired result. If $\delta < 0$ then the result follows by first taking $q \in \mathbb{Z}_0^+$ such that $t - qT$ is negative and then writing $t - qT$ as a (nonnegative) integer multiple of δ plus a remainder less than ϵ. □

Proof of Lemma $2'$: From the definition of ΔT we have $T = p \cdot \Delta T$ and $T' = q \cdot \Delta T$. Therefore, $iT + jT' = \Delta T(ip + jq)$ and it remains to be shown that

$$\{ip + jq | i \in \mathbb{Z}, j \in \mathbb{Z}_0^+\} = \mathbb{Z}. \tag{23}$$

Any integer multiple of q can be written as an integer multiple of p plus a remainder less than p, i.e., for each $j \in \mathbb{Z}_0^+$

$$jq = s_j p + r_j, \qquad s_j, r_j \in \mathbb{Z}_0^+, \qquad r_j < p.$$

The remainders $r_0, r_1, \cdots, r_{p-1}$ are all different because otherwise, if two of them (with indices $j_1 \neq j_2$ say) were identical, we would have $(j_1 - j_2)q = (s_{j_1} - s_{j_2})p$ and $|j_1 - j_2| < p$, which contradicts the assumption that p and q are coprime. Hence the set $\{r_0, r_1, \cdots, r_{p-1}\}$ equals the set $\{0, 1, 2, \cdots, p-1\}$. As a consequence $\{r_0 + p\mathbb{Z}, r_1 + p\mathbb{Z}, \cdots, r_{p-1} + p\mathbb{Z}\} = \mathbb{Z}$. From this we conclude that the set on the left-hand side of (23) contains $\mathbb{Z}$. It is obviously also contained in $\mathbb{Z}$. □

Proof of Lemma 3: We denote the integral appearing in (16) by L_i. Then for each $i \in \mathbb{Z}$ we have

$$\begin{aligned} L_{i+1} - L_i &= \int_{t'+iT}^{t'+(i+1)T} \xi^T e^{-F\tau} G \, d\tau \\ &= \int_{t'}^{t'+T} \xi^T (e^{-FT})^i e^{-Fs} G \, ds \\ &= \int_{t'}^{t'+T} \xi^T \sum_{j=0}^{n-1} \alpha_{j,i} (e^{-FT})^j e^{-Fs} G \, ds \\ &= \sum_{j=0}^{n-1} \alpha_{j,i} (L_{j+1} - L_j) \end{aligned} \tag{24}$$

where we used the substitution $\tau = iT + s$ in the first place, then the fact that, by the Cayley–Hamilton Theorem, each power of $(e^{-FT})^i, i \in \mathbb{Z}$, is a linear combination of $(e^{-FT})^j, j = 0, 1, 2, \cdots, n-1$, and finally we expressed the resulting integral in terms of $L_{j+1} - L_j = 0$ by using the second of the above equalities.

Since $L_{j+1} - L_j = 0$ for $j = 0, 1, 2, \cdots, n-1$ by assumption, the result $L_i = 0$ for all $i \in \mathbb{Z}$ follows recursively from (24). □

Proof of the Corollary—Observability: The proof is by contradiction. Suppose that for some $k \in \mathbb{Z}_0^+$ the state x_k cannot uniquely be determined from the measurement interval $[t_k, t_{k+(N+1)n-1}]$. Then there exists a nonzero vector $\xi \in \mathbb{R}^n$ such that

$$\begin{aligned} &He^{F(t_{k+(N+1)j+i} - t_k)} \xi \\ &\quad = 0, \quad \text{for all } j \in \{0, 1, \cdots, n-1\} \\ &\qquad\qquad i \in \{0, 1, \cdots, N\}. \end{aligned} \tag{25}$$

Since the sampling pattern is periodic, i.e., $t_{s+(N+1)} - t_s = NT + T'$ for all $s \in \mathbb{Z}_0^+$, we have

$$t_{k+(N+1)j+i} = t_{k+i} + (NT + T')j \tag{26}$$

and can rewrite (25) as

$$H[e^{F(NT+T')}]^j e^{F(t_{k+i} - t_k)} \xi = 0. \tag{27}$$

By the Cayley–Hamilton Theorem, (27) extends from $j \in \{0, 1, \cdots, n-1\}$ to all $j \in \mathbb{Z}$. Because this is true for all $i \in \{0, 1, \cdots, N\}$, we conclude that $He^{F(t_{k+\ell} - t_k)} \xi = 0$ for all $\ell \in \mathbb{Z}_0^+$ and hence that Σ_D is not observable. By the Theorem of Section II, this contradicts the assumption that Σ_C is observable.

Controllability: Suppose that for some $k \in \mathbb{Z}_0^+$ and $x', x'' \in \mathbb{R}^n$ there does not exist a control sequence $\{u_k, u_{k+1}, \cdots, u_{k+(N+1)n-1}\}$, which transfers the state from $x_k = x'$ to $x_{k+(N+1)n} = x''$.

We denote $k'' := k + (N+1)n$. Since we have

$$e^{-Ft_{k''}} x_{k''} + e^{-Ft_k} x_k = \sum_{s=k}^{k''-1} \int_{t_s}^{t_{s+1}} e^{-F\tau} G \, d\tau u_s, \tag{28}$$

it follows that a nonzero $\xi \in \mathbb{R}^n$ exists such that

$$\int_{t_s}^{t_{s+1}} \xi^T e^{-F\tau} G \, d\tau = 0 \tag{29}$$

for all $s \in \{k, k+1, \cdots, k+(N+1)n-1\}$. The set of all s, which are considered, can equivalently be represented by $\{k+(N+1)j+i | j \in \{0, 1, \cdots, n-1\}, i \in \{0, 1, \cdots, N\}\}$. This together with (26) and the substitution $\tau = \tau' + (NT + T')j$ allows us to rewrite (29) in the form

$$\int_{t_{k+i}}^{t_{k+i+1}} \xi^T [e^{-F(NT+T')}]^j e^{-F\tau'} G \, d\tau' = 0. \tag{30}$$

By the Cayley–Hamilton Theorem (30) extends from $j \in \{0, 1, \cdots, n-1\}$ to all $j \in \mathbb{Z}$. Since this is true for all $i \in \{0, 1, \cdots, N\}$, we conclude that $\int_{t_{k+\ell}}^{t_{k+\ell+1}} \xi^T e^{-F\tau} G \, d\tau = 0$ for all $\ell \in \mathbb{Z}_0^+$, and hence that Σ_D is not controllable. By the Theorem of Section II, this contradicts the assumption that Σ_C is controllable. □

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