Opinion Dynamics for Utility Maximizing Agents: Exploring the Impact of the Resource Penalty

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***Abstract*— We propose a continuous-time nonlinear model of opinion dynamics with utility-maximizing agents connected via a social influence network. A distinguishing feature of the proposed model is the inclusion of an opinion-dependent resource-penalty term in the utilities, which limits the agents from holding opinions of large magnitude. This model is applicable in scenarios where the opinions pertain to the usage of resources, such as money, time, computational resources etc. Each agent myopically seeks to maximize its utility by revising its opinion in the gradient ascent direction of its utility function, thus leading to the proposed opinion dynamics. We show that, for any arbitrary social influence network, opinions are ultimately bounded. For networks with *weak antagonistic relations*, we show that there exists a globally exponentially stable equilibrium using contraction theory. We establish conditions for the existence of consensus equilibrium and analyze the relative dominance of the agents at consensus. We also conduct a game-theoretic analysis of the underlying opinion formation game, including on Nash equilibria and on prices of anarchy in terms of *satisfaction ratios*. Additionally, we also investigate the oscillatory behavior of opinions in a two-agent scenario. Finally, simulations illustrate our findings.**

***Index Terms*— Opinion dynamics, Multi-agent systems, Utility maximization, Game theory.**

I. Introduction

OPINION dynamics deals with the modeling and mathematical analysis of how beliefs and ideas evolve and spread within social groups or networks over time. As collective opinions have far-reaching implications in diverse sectors such as policy, public health, sociology, finance or economics, it is paramount to understand their drivers and consequences. Even for smaller groups, comprehending opinion forming is a necessary first step toward the management of mixed multi-agent groups, their decision making and subsequent dynamic interactions. While many simpler opinion dynamic models have been developed, modeling and analysis of networked rational agents who react to neighbors' influence remains limited. In this work, we pay particular attention to the setting where the opinions of the agents are related to the usage of resources for a particular task. Each agent has heterogeneous resources available to them which limits their opinions about resource usage. We capture this in our model by including a resource penalty term in the utility functions.

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Literature Review: Many developments in the opinion dynamics literature stem from classical models, including the French-DeGroot (FD) model [1], [2], the Abelson model [3], the Taylor model [4], the Friedkin-Johnsen (FJ) model [5], the Hegselmann-Krause (HK) model [6], the Altafini model [7], and the Deffuant-Weisbuch (DW) model [8].

For an in-depth summary of the above models and survey of the literature, we refer the readers to [9]–[12]. These fundamental models serve as the foundation for many others in the literature. Some examples of recent models include the discrete-time versions of the Altafini model with time-varying signed networks [13], [14]. The work [15] proposes the *affine boomerang* opinion dynamics model with asynchronous opinion updating, incorporating the phenomenon of the *boomerang* effect into the dynamics. Unlike [7], [13], [14], in this model [15], the opinions do not converge to zero if the network does not satisfy a *structural balance* property, but rather exhibits bounded fluctuations. Reference [16] proposes the *expressed-private-opinion* model in which every agent's private opinion is influenced by the expressed opinions of its neighbors. The concept of *social power* was first introduced in [1] in order to identify the most influential agent in the social network. The DeGroot-Friedkin (DF) model [17] integrates the opinion formation process with social power evolution through a reflected self-appraisal mechanism. A generalized DF model was proposed in [18] and an extension of the DF model to stubborn agents was proposed in [19]. Some works on opinion dynamics [15], [20] also analyze fluctuations and periodic behavior of opinions.

The modeling of opinion dynamics through a game-theoretic or a utility maximization approach is still in its early stages, with only a limited number of works published thus far. Reference [21] showed that cooperative control problems, such as consensus seeking, can be effectively tackled using game-theoretic methods, particularly through potential and weakly acyclic games. The works [22], [23] interpret the FJ-model as best response dynamics within a game-theoretic framework, where each agent aims to minimize a cost function. The work [24] introduced a general framework for social influence grounded in the psychological concept of cognitive dissonance and demonstrated that various opinion dynamics models can be viewed as best response dynamics in a coordination game, where utilities are determined by dissonance functions. In a recent study [25], a dynamic influence maximization game is explored, where a set of competing *players* allocate their fixed resources over certain *individuals* (who hold opinions about *players*) to maximize their utilities in the long term. Reference [26] performs a game-theoretical analysis of the asynchronous HK model. The works [27], [28] capture the co-evolution of opinions and actions taken by the agents under a game-theoretic framework. The work [29] utilizes a continuous-time non-linear opinion dynamics model to tune the mutual cooperative behavior of agents in a repeated game. Within this framework, agents make strategic decisions relying upon rationality and reciprocity. Another approach, detailed in [30], introduces a discrete-time opinion dynamics model with a game-theoretic structure where the agents incur a combination of conformity and manipulation costs based on the opinions. The aim of each agent is to minimize this cost.

Exploration of how agents' resources impact their opinions and social influencing capabilities is currently an open question in the opinion dynamics literature. Motivated by this, we adopt a utility maximization and game-theoretic framework in the current work to investigate the limiting effects of agents' resources on their opinions about its usage. A preliminary version of this work appeared in [31], where we assumed the underlying social influence network to be complete. The current work extends [31] to the case of any general social network topologies and also allows for pairwise antagonistic relationships among the agents. We also investigate several new questions, such as exponential stability of the equilibria, price of anarchy and periodic evolution of opinions.

Contributions: The main contributions of this work are:

1) We define agent's utility function to capture the tradeoffs of internal opinion preferences, attachment (or stubbornness) toward its own opinion, conformity or non-conformity and lastly a resource penalty term. The resource penalty term in the utility function depends on the agent's resources, which is a representation of the agent's wealth, time etc. In this work, we assume that agents are involved in the opinion formation process to make some decisions about the use of their resources for a particular task. In the literature, most models of opinion dynamics are generic and seem to assume that the topic of discussion itself has no effect on the opinions' evolution. However, this is not realistic in our setting. Thus, we include a resource penalty term in the utility that keeps the opinions bounded. We propose a novel opinion dynamics model from the utility functions based upon the assumption that every agent myopically seeks to maximize its own utility. We refer to the underlying game as the *opinion formation game*.

2) Unlike the existing works which consider stubborn agents, under the proposed dynamics agents can reach consensus even if their internal preferences are different. If the agents' opinions reach a consensus equilibrium, we can use the *consensus dominance weights* of the agents to deduce the relative *dominance* of each agent. The consensus dominance weights depend on the resources of the agents.

3) We conduct a game-theoretic analysis of the opinion formation game. Our Nash equilibrium conditions hold even when the network consists of *antagonistic* relationships where opinions of some agents negatively influence the opinions of others. We provide a relation between the set of local Nash equilibria $\mathcal{NE}_l$, set of Nash equilibria $\mathcal{NE}$ of the opinion formation game and the set of equilibrium points $\mathcal{E}$ of the opinion dynamics. Specifically, we show that $\mathcal{NE} \subseteq \mathcal{NE}_l \subseteq \mathcal{E}$. Further, we give a condition for these sets to coincide.

4) In the case of *weak antagonistic* relationships, we show that the game exhibits a unique Nash equilibrium and the agents' opinions converge to it under the proposed dynamics starting from any arbitrary initial opinion profile. A special case of *weak antagonistic relationships* is one where antagonistic relationships are absent in the social network.

5) In the case where antagonistic relationships are absent, we bound the Price of Anarchy in terms of the *satisfaction ratios* to quantify the inefficiency of the unique Nash equilibrium. The satisfaction ratio of an agent at a given opinion profile of all agents is the ratio of the utility received by the agent at this opinion profile to the maximum possible utility; thus quantifying how "satisfied" that agent is with that opinion profile. Using these bounds, we show that if agents opinions converge to a consensus then it is a socially optimal outcome.

6) We analyze oscillatory and periodic opinion behavior. We show that it is necessary for the agents to have sufficiently strong antagonistic relations for the two-agent dynamics to have periodic solutions. We also provide sufficient conditions for a Hopf bifurcation to exist for the two-agent dynamics.

Notation: Throughout the paper, we use non-bold letters for denoting scalars, bold lowercase letters for denoting vectors, and bold uppercase letters for denoting matrices. The sets of natural numbers, real numbers, non-negative real numbers and positive real numbers are denoted by $\mathbb{N}, \mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$, respectively. Let $\mathbf{1} \in \mathbb{R}^n$ and $\mathbf{0} \in \mathbb{R}^n$ denote a vector with all elements equal to one and zero, respectively. For any vector $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{z}^\top$ denotes its transpose. For any scalar $a \in \mathbb{R}$, $|a|$ denotes its absolute value. For a set $\mathcal{S}$, $\mathcal{S}^n$ denotes the Cartesian product of $\mathcal{S}$ with itself n times. The empty set is denoted by $\varnothing$. We denote the difference of any two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ by $\mathcal{S}_1 \setminus \mathcal{S}_2$. Let $\mathsf{G} := (\mathcal{V}, \mathcal{L})$ be a directed graph, where $\mathcal{V}$ is the set of nodes and $\mathcal{L}$ is the set of directed arcs. In a directed graph G, a *directed walk* from a node $i_1 \in \mathcal{V}$ to any node $i_l \in \mathcal{V}$ is a sequence of nodes $i_1 \mapsto i_2 \mapsto \ldots \mapsto i_l$ such that $(i_s, i_{s+1}) \in \mathcal{L}$, $\forall s \in \{1, 2, \ldots, l-1\}$.

Organization of the paper: The rest of the paper is organized as follows. Section II introduces essential preliminaries for subsequent analysis. Section III presents the model for utility functions, and the opinion dynamics, as well as outlines the objectives of the paper. Section IV includes the asymptotic analysis of the model. Section V analyzes consensus equilibria, Nash equilibria and price of anarchy of the opinion formation game. Section VI deals with oscillatory behavior of two-agent opinion dynamics. Section VII includes simulations demonstrating our results. Finally, we conclude in Section VIII. We have included some of the longer proofs in the appendix.

II. PRELIMINARIES

Here, we recall some useful concepts from contraction theory. For a comprehensive description and a compilation of results on contracting dynamical systems, we refer the reader to [32]. First, we define the *induced logarithmic norm* of a matrix and give an interpretation of the same.

Definition 2.1: (**Logarithmic norm of a matrix** [32]) Given an induced matrix norm $\|.\|$ the induced log-norm of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is given by

$$\mu(\mathbf{A}) := \lim_{h\to 0^+} \frac{\|\mathbf{I}_n + h\mathbf{A}\| - 1}{h} \in \mathbb{R}.$$

If we use the induced $\infty-$matrix norm in the above definition, then we get the induced $\infty-$log norm of $\mathbf{A}$ as

$$\mu_\infty(\mathbf{A}) = \max_{i\in\mathcal{V}} \Big(a_{ii} + \sum_{j\in\mathcal{V}\setminus\{i\}} |a_{ij}| \Big). \qquad \bullet$$

The log-norm can be interpreted as the directional derivative of the matrix norm in the direction of $\mathbf{A}$ evaluated at the identity matrix $\mathbf{I}_n$. It should be noted that the induced log-norm of a matrix is not a matrix norm and can even be negative. This induced log-norm helps us in getting bounds on the norm of the solutions of a continuous time system. For example, consider a continuous-time homogeneous LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Using Coppel's inequalities [33], it can be shown that,

$$\|\mathbf{x}(t)\| \le e^{\mu(\mathbf{A})t}\|\mathbf{x}(0)\|.$$

If $\mu(\mathbf{A}) < 0$ then the above inequality implies that the solution $\mathbf{x}(t)$ converges to zero exponentially. We can also extend this idea to nonlinear systems. With this motivation, we define a *strongly contracting vector field.*

Definition 2.2: (***Strongly contracting vector field*** [32]) Let $\mathcal{C} \subset \mathbb{R}^n$ be convex. Let $\mathbf{f} : \mathcal{C} \to \mathbb{R}^n$ be differentiable and let $\mathbf{J}(\mathbf{z})$ denote its Jacobian. Then the vector field $\mathbf{f}$ is said to be *strongly infinitesimally contracting* on $\mathcal{C}$ with rate $\alpha > 0$ if

$$\mu(\mathbf{J}(\mathbf{z})) < -\alpha \,,\ \forall\, \mathbf{z} \in \mathcal{C}. \qquad \bullet$$

Finally, we recall a result that ensures the existence and uniqueness of an exponentially stable equilibrium for a strongly contracting system. The proof of this theorem is based on the Banach contraction theorem [32].

Theorem 2.3: (***Equilibrium of a strongly contracting system*** [32]) Suppose $\mathcal{C} \subset \mathbb{R}^n$ is convex, closed and positive $\mathbf{f}$-invariant. If $\mathbf{f} : \mathcal{C} \to \mathbb{R}^n$ is strongly infinitesimally contracting on $\mathcal{C}$ with rate $\alpha > 0$ then, $\mathbf{f}$ has a unique globally exponentially stable equilibrium $\mathbf{z}^* \in \mathcal{C}$ with global Lyapunov functions $V(\mathbf{z}) = \|\mathbf{z} - \mathbf{z}^*\|$ and $V(\mathbf{z}) = \|\mathbf{f}(\mathbf{z})\|$. $\bullet$

III. Modeling and Problem Setup

Consider a set $\mathcal{V} := \{1, \dots, n\}$ of n agents, with heterogeneous resources, that form opinions on a single topic, which is about the quantity of resources to use for a certain purpose. We aim to study the evolution of these opinions as the agents interact with each other over a social network. We start by defining utility functions for each agent. Then, we discuss the utility function and our motivations behind choosing it. Then, we derive the proposed opinion dynamics from the utility functions, assuming that each agent myopically seeks to maximize its utility. Thus, the coupling in the utility functions creates the coupling in the opinion dynamics.

Opinions, Utility Function and its Parameters: We denote the *expressed opinion* of agent $i \in \mathcal{V}$ at time t on the topic as $z_i(t) \in \mathbb{R}$. For brevity, we omit the time argument wherever there is no confusion. The vector $\mathbf{z} := [z_1, \cdots, z_n]^\top \in \mathbb{R}^n$ represents the stacked opinions z_i of all agents $i \in \mathcal{V}$. We first present the *utility function* of agent i, then describe the various parameters and provide motivation for the chosen structure.

The complete opinion profile $\mathbf{z} \in \mathbb{R}^n$ determines the utility for each agent $i \in \mathcal{V}$ as follows

$$U_i(\mathbf{z}, p_i) = -\frac{w_i}{2}\big(z_i - p_i\big)^2 - \sum_{k\in\mathcal{V}} \frac{a_{ik}}{2}\,(z_k - z_i)^2 - \frac{1}{4r_i} z_i^4, \quad (1)$$

where $z_i \in \mathbb{R}$ and $p_i \in \mathbb{R}$ are the expressed opinion and internal *preference* on the topic of agent $i \in \mathcal{V}$ respectively. The parameter $w_i \in \mathbb{R}_{>0}$ is the importance that agent $i \in \mathcal{V}$ attaches to its internal preference on the topic, while $r_i \in \mathbb{R}_{>0}$ is the resources available to agent $i \in \mathcal{V}$ and $a_{ik} \in \mathbb{R}$ is the weight of the influence of agent $k \in \mathcal{V}$ on agent $i \in \mathcal{V}$. We call the three terms in the utility function as the *preference term*, *social term* and *resource penalty*, respectively. We make the following standing assumption about the parameters in (1).

(SA1) ***(Parameters' signs.)*** For each agent $i \in \mathcal{V}$, $p_i \in \mathbb{R}$, $w_i \in \mathbb{R}_{>0}$, $r_i \in \mathbb{R}_{>0}$ and $a_{ik} \in \mathbb{R}$, $\forall k \in \mathcal{V}$. $\bullet$

Note that we allow for heterogeneous agents where each agent can have different parameters in their utility function. Moreover, in this paper, we assume that an agent's utility is affected by others via a directed social influence graph $\mathsf{G} := \big(\mathcal{V}, \mathcal{L}, \mathbf{A}\big)$. The elements of the *adjacency matrix* $\mathbf{A}$ are denoted by $a_{ij} \in \mathbb{R}$. If $a_{ik} \neq 0$ then there exists a directed link from node $k \in \mathcal{V}$ to node $i \in \mathcal{V}$ with link weight a_{ik}. This denotes that agent k's opinion influences the opinion of agent i. The sign of a_{ik} denotes the type of influence relationship and its magnitude denotes the degree of influence. Using this idea, we give the following definition.

Definition 3.1: ***(Neighbor set of an agent.)*** For each agent $i \in \mathcal{V}$, the set $\mathcal{N}_i^e := \{k \in \mathcal{V} \setminus \{i\} \mid a_{ik} < 0\}$ denotes the set of its *enemies* and the set $\mathcal{N}_i^f := \{k \in \mathcal{V} \setminus \{i\} \mid a_{ik} > 0\}$ denotes the set of its *friends*. Further,

$$\mathcal{N}_i := \mathcal{N}_i^e \cup \mathcal{N}_i^f = \{k \in \mathcal{V} \setminus \{i\} \mid a_{ik} \neq 0\},$$

denotes the set of neighbors of agent $i \in \mathcal{V}$. $\bullet$

Note that the self loop weights a_{ii}, for any $i \in \mathcal{V}$, do not affect the utility of the agents. We may assume them to be zero without loss of generality. The self influence of the agents is captured by the preference term in the utility function.

Discussion About the Utility Function: Overall, we seek to explain opinion evolution in social networks from a utility maximization perspective. In the sequel, we derive opinion dynamics as a gradient ascent by the agents of their utility functions, with respect to their own opinions. The three terms in (1) represent penalties on agent i's expressed opinion z_i arising from three factors. The preference term penalizes opinion deviation from the internal preference p_i. This penalty is directly proportional to the importance weight (or stubbornness) w_i. The social term penalizes the agent's opinion for being far away from or close to its neighbors' opinions depending upon the type of influence relationships, i.e., on the sign of a_{ik}'s. These two terms in the utility function lead to the first two terms in the opinion dynamics (2), which is the well known Taylor's model of opinion dynamics [4], when only non-antagonistic relations are allowed. In the presence of antagonistic relations, the second term in (1) is motivated by the *boomerang effect* [34]. So, our proposed model differs from the literature primarily in the resource penalty in the utility function (1).

The motivation for the resource penalty comes from the observation that if the topic of discussion is about the resource usage for some purpose, then the resource limitations of the agents must also have an effect on the opinion evolution. In our model, the resource penalty in the utility function restricts agent i from holding opinions of larger magnitude. In order to better motivate our model, consider the following example.

Example 3.2: Consider n agents with heterogeneous resource limitations. Let $z_i(t)$, the opinion of agent $i \in \mathcal{V}$ at time t, be the maximum amount of resources agent i is willing to spend on buying a particular product, if it were to do the buying at time t. Agent i has an internal opinion about the value of the product, denoted by p_i. Further, w_i models the confidence of agent i in its internal opinion. In addition, the agents are also influenced by the opinions of others within their social neighborhood. For example, they might look at product reviews or inquire within their social circles, which can influence their opinion about the value of the product. The weights a_{ik}'s model the trust or confidence of agent i has in agent k's ability to correctly value the product. Alternatively, these weights could simply model the degree to which agent i seeks to mimic or disagree with agent k. However, its spending is ultimately constrained by its resource limitations, which motivates the resource penalty in (1). •

The psychological and social effects of resource constraints on decision making of the agents have been studied in various fields such as psychology, economics, transportation etc. We refer the readers to the papers [35], [36] and references therein for more details. For example, [35] discusses the effects of financial constraints on consumer behaviors from four different perspectives: *resource scarcity*, *choice restriction*, *social comparison* and *environmental uncertainity*.

Remark 3.3: (On the Resource Penalty.) The parameter r_i denotes the amount of the resources (such as wealth, time etc.) available to agent $i \in \mathcal{V}$. It could be a proxy for the maximum budget an agent has to spend, such as the amount of money available to buy a certain good, the amount of time or other resources available for doing a task etc. In our work, we assume that the amount of resource r_i is static in time. This is reasonable to assume in scenarios where the agents' opinions are about the usage of their resources itself. The agents actually use their resources only after its opinion converges or after a sufficiently long evolution time of its opinion. Thus, we can think of the resource penalty as a soft constraint on their opinions, which are in turn about how much resources they could expend for a good or a task. The greater the resources that agent i has, the larger the magnitude of the opinions it can hold.

In this work, we choose a quartic resource penalty function; however, a more general class of penalty functions is also acceptable. Mathematically, all that is needed for ensuring the boundedness of opinions is to choose a non-negative resource penalty term that dominates the other terms in the utility (1) for large enough $|z_i|$. •

The following is a standing assumption in this paper.

(SA2) ***(Preferences are fixed parameters.)*** For each agent $i \in \mathcal{V}$, $p_i \in \mathbb{R}$ is a fixed parameter. •

The results can easily be extended for the case where $p_i = z_i(0)$, for each agent $i \in \mathcal{V}$. For example, we can let $\mathbf{p} = \mathbf{z}(0)$ and consider $(\mathbf{z}, \mathbf{p})$ as the state variables with $\dot{\mathbf{p}} = 0$. Since this provides little additional value, we choose to think of $\mathbf{p}$ as a fixed parameter. In the case where $\mathbf{p} = \mathbf{z}(0)$, we can study the dependence of the opinion evolution on the initial opinions $\mathbf{z}(0)$ by carrying out a parametric study.

Opinion Dynamics: We assume that at each time instant, agent $i \in \mathcal{V}$ revises its opinion by doing a gradient ascent of its utility function U_i, given in (1), with respect to its own opinion z_i. Thus, for each $i \in \mathcal{V}$, we have

$$\dot{z}_i = -\, w_i[z_i - p_i] + \sum_{k \in \mathcal{V}} a_{ik}[z_k - z_i] - \frac{z_i^3}{r_i}. \tag{2}$$

We can rewrite (2) equivalently as

$$\dot{z}_i = f_i(\mathbf{z}, p_i) := S_i(z_i, p_i) + C_i(\mathbf{z}), \ \ \forall i \in \mathcal{V}, \tag{3a}$$

$$S_i(z_i, p_i) := -w_i\left[z_i - p_i\right] - \frac{z_i^3}{r_i}, \tag{3b}$$

$$C_i(\mathbf{z}) := \sum_{k \in \mathcal{V}} a_{ik}[z_k - z_i]. \tag{3c}$$

Note that $\forall i \in \mathcal{V}$, the *self function* $S_i(\cdot, p_i)$ depends only on i's own opinion, its preference p_i and other parameters. On the other hand, $\forall i \in \mathcal{V}$, the *crowd function* $C_i(\cdot)$ depends on the deviations of i's opinion z_i from its neighbors' opinions z_k's. If $a_{ik} > 0$ (< 0) then $i \in \mathcal{V}$ wants to agree (disagree) with agent k and hence the term in $C_i(\cdot)$ corresponding to agent k drives z_i towards (away from) the opinion of $k \in \mathcal{V}$.

Now, we make some observations about the self function $S_i(\cdot, p_i)$, which hold $\forall i \in \mathcal{V}$. $S_i(\cdot, p_i)$ is continuous and strictly decreasing with $\lim_{z_i \to -\infty} S_i(z_i, p_i) = \infty$ and $\lim_{z_i \to +\infty} S_i(z_i, p_i) = -\infty$. Thus, $S_i(\cdot, p_i)$ has exactly one real root for every fixed value of $p_i \in \mathbb{R}$. Let us denote the real root of $S_i(\cdot, p_i)$ as $m_i(p_i) \in \mathbb{R}$, *i.e.*, $S_i(m_i(p_i), p_i) = 0$. Hereafter, we will exclude the preference argument in $S_i(\cdot, p_i)$ and $m_i(\cdot)$ wherever there is no confusion. Moreover, by considering $S_i(0)$ and $S_i(p_i)$, we can verify that $0 \leq |m_i| \leq |p_i|$ and $m_i p_i \geq 0$. $m_i \in \mathbb{R}$ can be interpreted as the opinion that agent i would attain under (2) if it were *socially closed* from the influence of other agents, *i.e.* $a_{ik} = 0$, $\forall k \in \mathcal{V}$.

Opinion Dynamics in Absence of Antagonistic Relations: An important special case is one where there are no antagonistic relations, which we formally state in the following assumption.

(A3) ***(No antagonistic relations.)*** $\forall i \in \mathcal{V}$, $\mathcal{N}_i^e = \varnothing$. •

Under Assumption **(A3)**, opinion dynamics (3a) reduces to

$$\dot{z}_i = f_i(\mathbf{z}, p_i) := S_i(z_i) + C_i^+(\mathbf{z}), \ \ \forall i \in \mathcal{V}, \tag{4}$$

$$\text{where} \ \ C_i^+(\mathbf{z}) = \begin{cases} 0 & ; \ \text{if } \mathcal{N}_i = \varnothing, \\ \displaystyle\sum_{j \in \mathcal{V}} a_{ij}\left[\bar{z}_i - z_i\right] & ; \ \text{otherwise} \end{cases} \tag{5}$$

$$\bar{z}_i := \sum_{k \in \mathcal{V}} \frac{a_{ik}}{\left(\sum_{j \in \mathcal{V}} a_{ij}\right)} z_k \,. \tag{6}$$

Note that, under Assumption **(A3)**, for any $i \in \mathcal{V}$, if $\mathcal{N}_i \neq \varnothing$ then $\bar{z}_i$ in (6) is well defined. Moreover, notice that this $\bar{z}_i$ is a convex combination of the opinions z_k of agent i's neighbors. So in (4), it suffices $\forall i \in \mathcal{V}$ to know only $\bar{z}_i$ and be unaware of other individual agents' opinions. Finally, note that $\forall i \in \mathcal{V}$,

$$S_i(z_i) \begin{cases} > 0, & z_i < m_i \\ = 0, & z_i = m_i \\ < 0, & z_i > m_i \end{cases}, \ C_i^+(\mathbf{z}) \begin{cases} > 0, & z_i < \bar{z}_i \\ = 0, & z_i = \bar{z}_i \\ < 0, & z_i > \bar{z}_i. \end{cases} \tag{7}$$

Objectives: The important analytical questions regarding the opinion dynamics (2) that we study in this paper include

- asymptotic properties
- existence of a consensus equilibrium and its properties
- characterization of Nash equilibria and their relation to the equilibria of the dynamics (2)
- bounds on price of anarchy
- oscillatory behavior of opinions in the two agent case.

We also study these questions for the cases outlined in Assumption **(A4)** and **(A3)** to provide stronger results.

IV. Asymptotic Behavior of Opinions

In this section, we study the asymptotic behavior of opinions. We show ultimate boundedness of opinions and provide conditions that guarantee existence and uniqueness of a globally exponentially stable equilibrium point. We denote the set of equilibrium points of (2) as a function of preferences $\mathbf{p}$ as

$$\mathcal{E}(\mathbf{p}) := \{\mathbf{z} \in \mathbb{R}^n \mid \dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mathbf{p}) = \mathbf{0}\}. \tag{8}$$

Hereafter, we drop the preference argument in $\mathcal{E}(\mathbf{p})$ wherever there is no confusion. We first guarantee that, under the dynamics (2), irrespective of the initial opinion profile, the opinions never grow unbounded and are in fact, ultimately bounded. The ultimate boundedness of opinions is a consequence of the resource penalty term in (1), which heavily penalizes any agent for holding opinions of greater magnitude.

Theorem 4.1: ***(Ultimate boundedness of opinions.)*** Let $\mathbf{z}(t)$ be the solution to (2) from the initial condition $\mathbf{z}(0)$. Then $\exists\, \eta \geq 0$ (independent of $\mathbf{z}(0)$) and $\exists\, T(\mathbf{z}(0)) \geq 0$ such that $|z_i(t)| \leq \eta$, $\forall i \in \mathcal{V}$, $\forall t \geq T$. Additionally, $\exists\, \Omega \subseteq \mathbb{R}^n$ which is convex, compact and positive invariant under (2). •

The existence of an ultimate bound can be shown using similar arguments involved in the proof of [31, Theorem 3.1]. Using the same arguments, we can also show existence of a convex, compact sublevel set Ω of the quadratic Lyapunov function $V(\mathbf{z}) := 0.5\, \mathbf{z}^\top \mathbf{z}$ such that $\dot{V}(\mathbf{z}) < 0$ outside Ω and the solution with any initial condition $\mathbf{z}(0) \in \mathbb{R}^n$ enters the set Ω in finite time $\bar{T}(\mathbf{z}(0))$. We skip the proof for brevity.

Note that one can explicitly find an ultimate bound η but we skip it for brevity. Also note that Theorem 4.1 holds for all values of the parameters and hence does not depend on the type of interactions among agents or the structure of the social network. From the proof of Theorem 4.1, we note that for any initial condition, the solution is uniformly bounded over all time. Then the local Lipschitzness of the vector field in (2) can be used to show existence and uniqueness of solutions of (2) for all time. Next, we characterize the conditions under which the opinions converge to an equilibrium. Recall that for each $i \in \mathcal{V}$, $S_i(\cdot)$ has a unique root at m_i, *i.e.* $S_i(m_i) = 0$. Using this, we show that the opinion of a socially closed agent (if one exists) converges to its corresponding m_i. The next result can be shown using (7) and the fact that the dynamics (2) for every such agent reduces to the scalar differential equation $\dot{z}_i = S(z_i)$. We skip the proof for brevity.

Lemma 4.2: ***(Convergence of opinions of socially closed agents.)*** Consider the dynamics given by (3a). Let $\mathbf{z}(t)$ denote the solution of (2) from an initial condition $\mathbf{z}(0)$. Then, $\lim_{t\to\infty} z_i(t) = m_i\, ;\, \forall i \in \mathcal{V}_{cl}$, where $\mathcal{V}_{cl}$ is the set of socially closed agents, i.e., $\mathcal{V}_{cl} := \{i \in \mathcal{V} \mid \mathcal{N}_i = \varnothing\} \subseteq \mathcal{V}$. •

We next address the general case, where there may be some socially closed agents and some agents influenced by others. In this case, the parameters, such as importance weights, resources and inter-agent influence weights, all play a role in determining whether there is an equilibrium point, its uniqueness and its stability. Using Theorem 2.3, we present a sufficient condition for (2) to have a unique globally exponentially stable equilibrium. We begin by stating an assumption.

(A4) ***(Weak antagonistic relations.)*** For each agent $i \in \mathcal{V}$, $w_i > \sum_{k \in \mathcal{N}_i^e} 2|a_{ik}|$, where $\mathcal{N}_i^e$ is the set of enemies of i. •

Notice that under Assumption **(SA1)**, $w_i > 0, \forall i \in \mathcal{V}$. Thus, the case of no antagonistic relations stated in Assumption **(A3)** is a special case of Assumption **(A4)**. All the results which hold under Assumption **(A4)** also hold under Assumption **(A3)**. Now, we present the main result of this section. Its proof is in the appendix.

Theorem 4.3: ***(Existence and uniqueness of equilibrium points.)*** Consider the dynamics given by (2). Suppose Assumption **(A4)** holds. Then, there exists a unique globally exponentially stable equilibrium point $\mathbf{z}^* \in \mathcal{E}$. •

Remark 4.4: ***(On the weak antagonistic relationships condition.)*** The condition of Assumption **(A4)** means that the *willingness* of every agent $i \in \mathcal{V}$ to hold an opinion close to its internal preference is greater than twice the aggregate influence weight of its enemies. Under this assumption, Theorem 4.3 shows that the opinions converge to a unique equilibrium. •

Based on Lemma 4.2 and Theorem 4.3, we can give a stronger result for ultimate boundedness and on the location of the unique equilibrium in case there are no antagonistic relations among agents. We deal with this case next.

Convergence of opinions in the absence of antagonistic relations: In the absence of antagonistic relationships among agents, it is possible to give an ultimate bound that is more intuitive and easier to compute using the unique roots of the self functions (m_i such that $S_i(m_i) = 0$, $i \in \mathcal{V}$). We can also say that the unique equilibrium in this case is in a specific set defined by m_i's, for $i \in \mathcal{V}$. Let,

$$m_{\min}(\mathbf{p}) := \min\{m_i\}_{i\in\mathcal{V}},\; m_{\max}(\mathbf{p}) := \max\{m_i\}_{i\in\mathcal{V}}, \tag{9}$$

and the corresponding interval

$$\mathcal{M}(\mathbf{p}) := [m_{\min}(\mathbf{p}), m_{\max}(\mathbf{p})]\,. \tag{10}$$

For brevity, we will exclude the preference argument in (9) and (10) wherever there is no confusion. We are now ready to show that the opinions converge to $\mathcal{M}^n$ (proof in Appendix).

Proposition 4.5: ***(Convergence to the set $\mathcal{M}^n$ in absence of antagonistic relations.)*** Consider the opinion dynamics given by (2). Suppose Assumption **(A3)** holds. Let $m_{\min}$, $m_{\max}$ and $\mathcal{M}$ be as defined in (9) and (10) respectively. Then, $\mathcal{M}^n$ is positively invariant under the opinion dynamics (2). Let $\mathbf{z}(t)$ be the solution to (2) from an initial condition $\mathbf{z}(0) \in \mathbb{R}^n$. Then $\mathbf{z}(t)$ converges to $\mathcal{M}^n$. Further, define $\mathcal{V}_{\max} := \{i \in \mathcal{V} \mid m_i = m_{\max}\}$ and $\mathcal{V}_{\min} := \{i \in \mathcal{V} \mid m_i = m_{\min}\}$. Suppose $m_{\min} < m_{\max}$. Then the following statements are equivalent.

(i) The unique equilibrium $\mathbf{z}^* \in \mathcal{E}$ lies in the interior of $\mathcal{M}^n$ and $\exists T(\mathbf{z}(0)) \geq 0$ such that $\mathbf{z}(t) \in \mathcal{M}^n$, $\forall t \geq T(\mathbf{z}(0))$.

(ii) $\forall i \in \mathcal{V}_{\max}$, $\exists$ a directed walk in G starting from $j \in \mathcal{V} \setminus \mathcal{V}_{\max}$ to i and $\forall i \in \mathcal{V}_{\min}$, $\exists$ a directed walk in G starting from $j \in \mathcal{V} \setminus \mathcal{V}_{\min}$ to i. •

Under Assumption **(A3)**, Proposition 4.5 guarantees that the set $\mathcal{M}^n$ is positively invariant. Moreover, the unique globally exponentially stable equilibrium point $\mathbf{z}^*$ that Theorem 4.3 guarantees lies in $\mathcal{M}^n$. Thus, we can immediately guarantee that $\mathbf{z}(t)$ converges to $\mathcal{M}^n$, possibly asymptotically. If $m_{\min} < m_{\max}$ then only agents belonging to sets $\mathcal{V}_{\max}$ and $\mathcal{V}_{\min}$ can have equilibrium opinions at the boundary of the set $\mathcal{M}$. For the unique equilibrium to be in the interior of $\mathcal{M}^n$, it is both necessary and sufficient that every agent in the sets $\mathcal{V}_{\max}$ and $\mathcal{V}_{\min}$ is directly or indirectly influenced by at least one agent in sets $\mathcal{V}\backslash\mathcal{V}_{\max}$ and $\mathcal{V}\backslash\mathcal{V}_{\min}$, respectively. Note that this condition is satisfied if the social network G is strongly connected. Finally, if the unique equilibrium $\mathbf{z}^*$ lies in the interior of $\mathcal{M}^n$ then solutions converge to $\mathcal{M}^n$ in finite time. In this case, the ultimate bound $\mathcal{M}^n$ has an additional advantage that it depends only on the m_i's (whose interpretation is provided in Section III) and hence can be computed easily using the parameters w_i, p_i and r_i. Finally, note that if $m_{\max} = m_{\min}$ then the opinions of agents in absence of antagonistic relations always converge to a unique consensus equilibrium.

Now, if the weak antagonistic relationships condition given in Assumption **(A4)** holds and there exists a consensus equilibrium, then, from Theorem 4.3, it is the only equilibrium of the dynamics and the agents always achieve consensus starting from any initial opinion vector. •

When the agents attain a consensus equilibrium, we can measure how much influence an agent has on the whole group by measuring the deviation of the consensus value from its preference. Note that if $p_i = 0$, for some $i \in \mathcal{V}$, then $m_i = 0$ and hence the only consensus equilibrium possible (if it exists) is neutral. So we consider $p_i \neq 0,\ \forall i \in \mathcal{V}$ to give the next result on dominance and discuss it in the remark following it.

***Proposition** 5.3: (**Consensus deviation from preference**.)* Consider the dynamics (2) or equivalently (3a). Suppose that $p_i \neq 0, \forall i \in \mathcal{V}$. For each agent $i \in \mathcal{V}$, let us define $\sigma_i := w_i r_i$ and $\Delta_i(\xi) := |p_i - \xi|$. If $\xi\mathbf{1} \in \mathcal{E}$, with $\xi \in \mathbb{R}$, then $\sigma_i \Delta_i(\xi) = \sigma_j \Delta_j(\xi)$, $\forall i, j \in \mathcal{V}$. In particular, $\sigma_i > \sigma_j$ iff $\Delta_i(\xi) < \Delta_j(\xi)$.

Proof: Since $p_i \neq 0,\ \forall i \in \mathcal{V}$, we also have $0 < |m_i| < |p_i|\ \forall i \in \mathcal{V}$. Then by Theorem 5.1, $\xi \neq 0$. Further, from Theorem 5.1, we get $r_i S_i(\xi) = 0,\ \forall i \in \mathcal{V}$, which then implies

$$\sigma_i(p_i - \xi) = \xi^3 = \sigma_j(p_j - \xi),\ \forall i, j \in \mathcal{V}.$$

Since $\sigma_i > 0,\ \forall i \in \mathcal{V}$, the result now follows. ■

***Remark** 5.4: (**Dominance in consensus**.)* Let us call $\forall i \in \mathcal{V}$, the scalar σ_i as the *consensus dominance* weight of the agent i. Suppose all agents have a non-neutral preference and they attain consensus at $\xi \in \mathbb{R}$. Then, Proposition 5.3 states that if an agent $i \in \mathcal{V}$ has higher consensus dominance weight than agent $j \in \mathcal{V}$, then ξ is closer to i's preference than that of j. Note that the consensus dominance weight is directly proportional to the importance weight an agent assigns to its preference and the resources available to it. Thus, an agent with very high resources can exert more influence even if it gives less importance weight to its internal preference. On the other hand, if an agent has lower resources, then it has to have much higher importance weight to have more influence in the group. •

Next, we study the relation between equilibria of the opinion dynamics (2) and Nash equilibria of the underlying game.

V. Consensus and Nash Equilibria

In this section, we analyze consensus equilibria of the opinion dynamics (2) and Nash equilibria of the underlying game. We also explore the relation between the Nash equilibrium set of the underlying game and the set of equilibria of the opinion dynamics (2). Finally, we also analyze the price of anarchy.

A. Consensus Equilibria

First, we deal with the consensus equilibria of the model, i.e., equilibria of the form $\xi\mathbf{1}$, with $\xi \in \mathbb{R}$. We refer to the case of $\xi = 0$ as a *neutral consensus* since all the agents have neutral opinions (equal to 0) in this case. On the other hand, we refer to the case of $\xi \neq 0$ as a non-neutral consensus. In the following lemma, we present conditions for (2) to have a consensus equilibrium. We use the form of the dynamics in (3a) and the functions in (3) to justify our claims.

***Theorem** 5.1: (**Necessary and sufficient conditions for existence of a consensus equilibrium**.)* Consider the dynamics (2) (equivalently (3a)). For each $i \in \mathcal{V}$, let $m_i \in \mathbb{R}$ be the unique point such that $S_i(m_i) = 0$. Then, $\xi\mathbf{1} \in \mathcal{E}$ if and only if $m_i = \xi$, $\forall i \in \mathcal{V}$.

Proof: If $z_i = \xi$, $\forall i \in \mathcal{V}$, for some $\xi \in \mathbb{R}$ then $C_i(\xi\mathbf{1}) = 0$, $\forall i \in \mathcal{V}$. Hence, from (3a), $\xi\mathbf{1} \in \mathcal{E}$ iff $S_i(\xi) = 0, \forall i \in \mathcal{V}$. Since m_i is the unique root of $S_i(.)$, the claim follows. ■

***Remark** 5.2: (**Consensus formation among agents**.)* Theorem 5.1 states that it is both necessary and sufficient for all the m_i's to be the same for the opinion dynamics model to have a consensus equilibrium. It is evident that if the agents are to arrive at a consensus equilibrium, then all their preferences p_i's must be of the same sign. When $p_i = 0$, $\forall i \in \mathcal{V}$, then the only possible consensus equilibrium is the neutral consensus, *i.e.*, every agent reaches a neutral opinion on the topic. If the preferences of the agents have different signs, then the opinions of agents can never reach an exact consensus in equilibrium. However, other equilibria that are arbitrarily close to consensus may still exist.

B. Nash Equilibria

Here we carry out a Nash Equilibrium analysis of the opinion formation game. Recall that every agent is interested in maximizing its utility U_i given in (1) by suitably choosing its opinion z_i. Thus, this gives a *strategic form game* $\mathcal{G} = \langle \mathcal{V}, (\mathbb{R})_{i\in\mathcal{V}}, (U_i)_{i\in\mathcal{V}} \rangle$ among the set of agents $\mathcal{V}$, with the *strategy* of agent $i \in \mathcal{V}$ being its opinion $z_i \in \mathbb{R}$ and its utility function being $U_i(\cdot)$. For the sake of convenience, we let $\mathbf{z}_{-i}$ denote the opinions of all agents other than i. Then, the set of *Nash equilibria* of the game $\mathcal{G}$ is

$$\mathcal{NE}(\mathbf{p}) := \{\mathbf{z}^* \in \mathbb{R}^n \mid \forall i \in \mathcal{V}, \\ U_i(z_i^*, \mathbf{z}_{-i}^*, p_i) \geq U_i(z_i, \mathbf{z}_{-i}^*, p_i), \forall z_i \in \mathbb{R}\}. \quad (11)$$

Note that for a Nash equilibrium $\mathbf{z}^*$, z_i^* is agent i's best response over all opinions $z_i \in \mathbb{R}$ to $\mathbf{z}_{-i}^*$, the opinion profile of all the other agents. However, in the dynamics (2), each agent updates its opinion according to the gradient ascent of its utility with respect to its opinion while assuming that the other agents do not change their opinions. Hence, the agents at each time instant revise their opinion only towards the "local" best response. This motivates the next definition.

Definition** 5.5: **(Local Nash equilibrium.) A strategy profile $\mathbf{z}^* \in \mathbb{R}^n$ is said to be a local Nash equilibrium if and only if $\forall i \in \mathcal{V}$, $\exists\ \rho_i \in \mathbb{R}_{>0}$ such that

$$U_i(z_i^*, \mathbf{z}_{-i}^*, p_i) \geq U_i(z_i, \mathbf{z}_{-i}^*, p_i),\ \forall\, z_i \text{ s.t. } |z_i^* - z_i| \leq \rho_i\,.$$

The set of local Nash equilibria of $\mathcal{G}$ is denoted by $\mathcal{NE}_l(\mathbf{p})$.•

For simplicity, we will exclude the preference arguments in $\mathcal{NE}(\mathbf{p})$ and $\mathcal{NE}_l(\mathbf{p})$ wherever there is no confusion. It is easy to see that a Nash equilibrium is also a local Nash equilibrium and hence $\mathcal{NE} \subseteq \mathcal{NE}_l$. In the next result, we show that every local Nash equilibrium is an equilibrium point of (2).

Lemma** 5.6: **(Local Nash equilibrium is an equilibrium of the opinion dynamics.) If an opinion profile $\mathbf{z}^*$ is such that $\mathbf{z}^* \in \mathcal{NE}_l$ then $\mathbf{z}^* \in \mathcal{E}$.

Proof: Since $\mathbf{z}^* \in \mathcal{NE}_l$, it implies that $\forall i \in \mathcal{V}$, z_i^* locally maximizes $U_i(\cdot, \mathbf{z}_{-i}^*)$. Thus, the partial derivative of $U_i(\cdot)$ with respect to z_i evaluated at $(z_i^*, \mathbf{z}_{-i}^*)$ is *zero*. The claim then follows immediately from (2) and (8). ■

Lemma 5.6 states that every local Nash equilibrium of $\mathcal{G}$ is also an equilibrium point of dynamics (2). But the converse need not be true. In the following result, we give conditions for an opinion profile $\mathbf{z}^* \in \mathbb{R}^n$ that is an equilibrium point of the opinion dynamics to be a local Nash equilibrium of the opinion formation game.

Theorem** 5.7: **(Conditions for an equilibrium point of** (2) **to be a local Nash equilibrium.) Consider the dynamics (2) and the set of equilibrium points $\mathcal{E}$ in (8). Let $\mathbf{z}^* = \left(z_i^*, \mathbf{z}_{-i}^*\right) \in \mathcal{E}$. Then, $\mathbf{z}^* \in \mathcal{NE}_l$ only if for each $i \in \mathcal{V}$,

$$(z_i^*)^2 \geq \tau_i := \frac{r_i}{3}\left[\sum_{k \in \mathcal{N}_i^e} |a_{ik}| - \sum_{k \in \mathcal{N}_i^f} a_{ik} - w_i\right]. \qquad (12)$$

Moreover, if inequality (12) is strict $\forall i \in \mathcal{V}$, then $\mathbf{z}^* \in \mathcal{NE}_l$.

Proof: Let the hypothesis be true. From Definition 5.5 we know that $\mathbf{z}^* = \left(z_i^*, \mathbf{z}_{-i}^*\right)$ is a local Nash equilibrium if and only if $\forall i \in \mathcal{V}$, z_i^* locally maximizes $U_i(\cdot, \mathbf{z}_{-i}^*)$. Now since $\mathbf{z}^* \in \mathcal{E}$, by the definitions in (2) and (8), it is clear that for each $i \in \mathcal{V}$, z_i^* satisfies the first-order necessary conditions for it to be a local maximizer of $U_i(\cdot, \mathbf{z}_{-i}^*)$.

Now, suppose that $\mathbf{z}^* \in \mathcal{NE}_l$. Then, we have that

$$\left.\frac{\partial^2}{\partial z_i^2} U_i(z_i, \mathbf{z}_{-i}^*)\right|_{z_i^*} = \frac{3}{r_i}\left[\tau_i - (z_i^*)^2\right] \leq 0, \quad \forall i \in \mathcal{V}\,. \quad (13)$$

This proves the necessary condition in (12). Finally, note that if (13) is a strict inequality for a $\mathbf{z}^* \in \mathcal{E}$, then by the second order sufficiency condition for a point to be a local maximizer, z_i^* is a local maximizer of $U_i(\cdot, \mathbf{z}_{-i}^*)$ for each $i \in \mathcal{V}$. ■

The statement of the previous result can be combined with the result in Theorem 4.1 to provide a condition for which the opinion formation game does not have any local Nash equilibrium. We state this in the next result, proof of which is intuitive since no equilibrium point of (2) can exist beyond any ultimate bound (which always exists).

Corollary** 5.8: **(Non-existence of local Nash equilibria.) Suppose $\eta_i > 0$ is an ultimate bound on z_i for any $i \in \mathcal{V}$ under (2). If there exists $i \in \mathcal{V}$ such that $\tau_i > \eta_i^2$, with τ_i defined in (12), then $\mathcal{NE}_l = \varnothing$. ■

Finally, to end this subsection, we provide a sufficient condition under which the different equilibria sets ($\mathcal{E}$, $\mathcal{NE}$ and $\mathcal{NE}_l$) are equal. Additionally, we also give a sufficient condition under which the opinion formation game has a unique Nash equilibrium. We state this in the following result.

Theorem** 5.9: **(Equality of equilibria sets and uniqueness of Nash equilibrium.) Consider the dynamics in (3a) and the set of equilibrium points $\mathcal{E}$ in (8). Suppose for each agent $i \in \mathcal{V}$, $\tau_i \leq 0$. Then, $\mathcal{NE} = \mathcal{NE}_l = \mathcal{E}$. Moreover, if Assumption **(A4)** holds then it is a singleton set.

Proof: Suppose that $\tau_i \leq 0$, $\forall i \in \mathcal{V}$. First, it is obvious that $\mathcal{NE} \subseteq \mathcal{NE}_l$ and Lemma 5.6 implies that $\mathcal{NE}_l \subseteq \mathcal{E}$. Next, we show that $\mathcal{E} \subseteq \mathcal{NE}_l$. Consider an equilibrium $\mathbf{z}^* \in \mathcal{E}$. It can be easily verified that for each $i \in \mathcal{V}$, $U_i(\cdot, \mathbf{z}_{-i}^*)$ is strictly concave in z_i. Since $\mathbf{z}^* \in \mathcal{E}$, the first order necessary condition for a point to be a local maximizer of $U_i(\cdot, \mathbf{z}_{-i}^*)$ is satisfied for each i. Hence, $\mathbf{z}^* \in \mathcal{NE}_l$. Finally, we show that $\mathcal{NE}_l \subseteq \mathcal{NE}$. Since $\forall i \in \mathcal{V}$, $U_i(\cdot, \mathbf{z}_{-i}^*)$ is a strictly concave function for each $\mathbf{z}_{-i}^*$, the implication follows directly from the definitions of $\mathcal{NE}$ and $\mathcal{NE}_l$. This completes the proof of the first part of the claim. Now suppose that Assumption **(A4)** holds. Then again $\tau_i < 0$, $\forall i \in \mathcal{V}$. Uniqueness of Nash equilibrium now follows from Theorem 4.3 and first part of the claim. ■

***Remark** 5.10:* We can interpret w_i as the weight of influence of a stubborn virtual agent on agent i. Using this interpretation, let us define the *weighted in-degree* for each agent $j \in \mathcal{V}$ in social network G as $d_j^{in} := \left[w_j + \sum_{k \in \mathcal{V} \setminus \{j\}} a_{jk}\right]$. Then for each $j \in \mathcal{V}$, $\tau_j = -r_j d_j^{in}/3$ and $\tau_j < 0 (= 0)$ if and only if $d_j^{in} > 0 (= 0)$. The condition $d_j^{in} > 0$, $\forall j \in \mathcal{V}$ can be interpreted as follows: For each agent $i \in \mathcal{V}$, the aggregate influence weight of its friends (including the virtual fully stubborn agent since $w_i > 0$) is greater than the aggregate influence weight of its enemies. If this holds true then Theorem 5.9 states that each equilibrium point of the dynamics is also a Nash equilibrium of the game. Next, if Assumption **(A4)** holds then for each $j \in \mathcal{V}$, $d_j^{in} > 0$ and hence $\tau_j < 0$. In other words, the opinion formation game has a unique Nash equilibrium if the *willingness* of each agent to be close to its internal preference (or stubbornness) is more than twice the aggregate influence weight of its enemies. Moreover, thanks to Theorem 4.3, we can say that the opinions under (2) always converge to this unique Nash equilibrium, starting from any initial condition. •

C. Price of Anarchy

Next, we analyze the *price of anarchy* of the game underlying the opinion dynamics (2). In the entirety of this subsection, we will use cost minimization perspective rather than utility maximization. We let the cost incurred by agent $i \in \mathcal{V}$ for opinion profile $\mathbf{z}$ be $\chi_i(\mathbf{z}, \mathbf{p}) := -U_i(\mathbf{z}, \mathbf{p})$, where $U_i(\mathbf{z}, \mathbf{p})$ is given by (1). In order to ensure non-negativity of the prices of anarchy defined below, we need $\chi_i(\mathbf{z}, \mathbf{p}) \geq 0\,;\ \forall \mathbf{z} \in \mathbb{R}^n$. Hence, in this subsection we will assume that all inter-agent relations are non-antagonistic, i.e., Assumption **(A3)** holds. We consider price of anarchy for two most commonly used objective functions in the game theory literature, namely, the *egalitarian* and the *utilitarian* costs

$$\mathsf{C}_E(\mathbf{z},\mathbf{p}) := \max_{i\in\mathcal{V}} \chi_i(\mathbf{z},\mathbf{p}), \quad \mathsf{C}_U(\mathbf{z},\mathbf{p}) := \sum_{i\in\mathcal{V}} \chi_i(\mathbf{z},\mathbf{p}),$$

to measure the inefficiency of Nash equilibria of the game underlying the opinion dynamics (2). Hereafter, we will again exclude the preference arguments in $\chi_i(\cdot,\cdot)$, $\mathsf{C}_E(\cdot,\cdot)$ and $\mathsf{C}_U(\cdot,\cdot)$ for brevity. In the following definitions of price of anarchy, we will assume $\mathbf{p} \neq \mathbf{0}$ to ensure the positivity of $\mathsf{C}_E(\mathbf{z})$ and $\mathsf{C}_U(\mathbf{z})$ at any $\mathbf{z} \in \mathbb{R}^n$. This ensures that prices of anarchy (14) are well defined. We discuss the case when $\mathbf{p} = \mathbf{0}$ in a later remark.

Definition 5.11: (Price of anarchy.) Consider the opinion formation game $\mathcal{G}$ corresponding to the opinion dynamics (2), and $\mathcal{NE}$, the set of its Nash equilibria. Suppose Assumption **(A3)** holds and $\mathbf{p} \neq \mathbf{0}$. The egalitarian and the utilitarian prices of anarchy, π_e and π_u, respectively are defined as

$$\pi_e := \frac{\max_{\mathbf{z}\in\mathcal{NE}} \mathsf{C}_E(\mathbf{z})}{\min_{\mathbf{z}\in\mathbb{R}^n} \mathsf{C}_E(\mathbf{z})} \geq 1; \quad \pi_u := \frac{\max_{\mathbf{z}\in\mathcal{NE}} \mathsf{C}_U(\mathbf{z})}{\min_{\mathbf{z}\in\mathbb{R}^n} \mathsf{C}_U(\mathbf{z})} \geq 1. \tag{14}$$

•

In Definition 5.11, π_e compares the cost incurred by the worst performing agent at the worst Nash equilibrium to the minimum possible cost C_E. Similarly, π_u compares the total cost incurred by all agents at the worst Nash equilibrium to the minimum possible cost C_U. The closer the value of the prices of anarchy (14) is to unity, the better the quality of the Nash equilibrium. We now define the *satisfaction ratio* at opinion profile $\mathbf{z}$ for every agent $i \in \mathcal{V}$ in order to compare the cost incurred by agent i at $\mathbf{z}$ and the minimum possible cost it could incur. Using these ratios, we can give some upper bounds on the prices of anarchy (14). In order to keep the satisfaction ratios well defined, we need positivity of $\chi_i(\mathbf{z})$ everywhere. Hence we assume the following,

(A5) ***(Non-zero preferences.)*** $\forall i \in \mathcal{V}$, $p_i \neq 0$. •

Definition 5.12: (Satisfaction ratio.) Consider the cost function $\chi_i(\mathbf{z})$ of each agent $i \in \mathcal{V}$. Suppose Assumptions **(A3)** and **(A5)** hold. The satisfaction ratio SR_i for any agent $i \in \mathcal{V}$ at any $\mathbf{z} \in \mathbb{R}^n$ is defined as,

$$\mathsf{SR}_i(\mathbf{z}) := \frac{\chi_i(\mathbf{z})}{\min_{\mathbf{z}\in\mathbb{R}^n} \chi_i(\mathbf{z})} \geq 1. \tag{15}$$

The next result shows that if there are no antagonistic relations among the agents, then for each agent $i \in \mathcal{V}$, $\chi_i(\cdot)$ is convex. Note that this result says $\chi_i(\mathbf{z})$ is convex in $\mathbf{z}$ and not just with respect to z_i. Proof of the following result is in the appendix.

Lemma 5.13: (Convexity of cost function.) Suppose that Assumption **(A3)** holds. Then for each $i \in \mathcal{V}$, $\chi_i(\cdot)$ is convex with $m_i\mathbf{1}$ as one of its minimizer. •

Now, from Theorem 4.3 and Theorem 5.9, we know that in the absence of antagonistic relations, the game has a unique Nash equilibrium $\mathbf{z}^* \in \mathcal{NE}$. In the next result, whose proof is in the Appendix, we use the satisfaction ratios to give an upper bound on π_e and π_u.

Theorem 5.14: (Bounds on prices of anarchy.) Consider the dynamics (2). Suppose Assumptions **(A3)** and **(A5)** hold. Let us denote the unique Nash equilibrium of the game underlying the opinion dynamics (2) by $\mathbf{z}^* \in \mathcal{NE}$. Then,

$$1 \leq \pi_e \leq \max_{i\in\mathcal{V}} \mathsf{SR}_i(\mathbf{z}^*) \ , \ 1 \leq \pi_u \leq \sum_{i\in\mathcal{V}} \mathsf{SR}_i(\mathbf{z}^*) \ , \tag{16}$$

with π_e and π_u as defined in Definition 5.11. •

A consequence of this result is that, for the special case where the unique Nash equilibrium is a consensus equilibrium, $\pi_e = \pi_u = 1$, i.e., consensus equilibrium (if it exists) is socially optimal. We formally state and prove this next.

Corollary 5.15: (PoA is unity for a non-neutral consensus equilibrium.) Consider the dynamics (2). Suppose Assumptions **(A3)** and **(A5)** hold. Suppose the unique $\mathbf{z}^* \in \mathcal{NE} = \mathcal{E}$ is a non-neutral consensus equilibrium, i.e., $\mathbf{z}^* = m\mathbf{1}$ for some $m \neq 0$. Then, $\pi_e = \pi_u = 1$.

Proof: From Theorem 5.1, Lemma 5.13 and Definition 5.12, it follows that, $\mathsf{SR}_i(m\mathbf{1}) = 1$, $\forall i \in \mathcal{V}$. From (16), we thus have $\pi_e = 1$. From the proof of Theorem 5.14 for the bound on π_u, we have

$$1 \leq \pi_u \leq \frac{\sum_{i\in\mathcal{V}} \chi_i(m\mathbf{1})}{\sum_{i\in\mathcal{V}} \chi_i(m\mathbf{1})} = 1\,,$$

since $\mathbf{z}^* = m\mathbf{1}$ and $m_i = m$, $\forall i \in \mathcal{V}$. ■

Note that neutral and non-neutral consensus equilibrium exists only if $\mathbf{p} = \mathbf{0}$ and Assumption **(A5)** holds, respectively. Consensus equilibrium cannot exist if $\exists i, j \in \mathcal{V}$ with $p_i = 0$ and $p_j \neq 0$. Corollary 5.15 states that non-neutral consensus is socially optimal. In the next remark, we argue that neutral consensus is also socially optimal.

Remark 5.16: (Neutral consensus is socially optimal.) Let Assumption **(A3)** hold and $\mathbf{p} = \mathbf{0}$. Theorems 5.1, 4.3 and 5.9 imply that the unique equilibrium point $\mathbf{z}^* = \mathbf{0} \in \mathcal{E} = \mathcal{NE}$. Now, from Lemma 5.13 we know that $\mathbf{0}$ is a minimizer of $\chi_i(\mathbf{z})$, $\mathsf{C}_E(\mathbf{z})$ and $\mathsf{C}_U(\mathbf{z})$. Thus, prices of anarchy (14) and satisfaction ratios (15) are not defined in this case. But since $\mathsf{C}_E(\mathbf{z}^*) = \min_{\mathbf{z}\in\mathbb{R}^n} \mathsf{C}_E(\mathbf{z}) = 0 = \mathsf{C}_U(\mathbf{z}^*) = \min_{\mathbf{z}\in\mathbb{R}^n} \mathsf{C}_U(\mathbf{z})$, neutral consensus equilibrium is also socially optimal. •

VI. Oscillatory Behavior of Opinions

In [31], we studied a special case of the dynamics proposed in the current paper. We showed through numerical simulations that in certain scenarios, the opinions exhibit oscillatory behavior. In this section, we analyze such behavior in the case of a pair of agents $\mathcal{V} = \{1, 2\}$. For a general social network with n agents, the analysis is significantly more complicated due to the higher dimension of the state space but also due to the complexity added by the social network. Thus, the analysis in the general case of a social network with n agents would have to be a separate research work in itself and is out of the scope of this paper. Even though our analysis is restricted to two agents, it is still relevant since it can help us understand opinion behaviors or decision making in important systems such as two-party politics, duopoly economic markets etc.

The opinion dynamics (2) for two agents is,

$$\dot{z}_1 = S_1(z_1) + a_{12}\big[z_2 - z_1\big], \ \dot{z}_2 = S_2(z_2) - a_{21}\big[z_2 - z_1\big]. \tag{17}$$

For the *two-agent dynamics* (17), it can be easily verified that if the opinions $z_1(t)$ and $z_2(t)$ exhibit oscillatory behavior then they will have the same fundamental period of oscillations. We state this in the following result (proof in Appendix).

Lemma 6.1: (***Equal period of oscillations***) Consider the two-agent dynamics (17). Suppose the opinions $z_1(t)$ and $z_2(t)$ exhibit oscillations with fundamental periods $T_1 > 0$ and $T_2 > 0$, respectively. Then, $T_1 = T_2$. •

Note that periodic orbits can exist for (17) only if at least one of the agents has an antagonist influence on the other as otherwise Theorem 4.3 guarantees convergence of opinions to the unique equilibrium from any arbitrary initial condition. In the next result (proof in Appendix), we give a necessary condition for the existence of periodic orbits for (17).

Lemma 6.2: (***Necessary condition for existence of periodic solutions***) Consider the two-agent dynamics (17). Opinions $z_1(t)$ and $z_2(t)$ exhibit periodic behavior only if all the following hold: (i) $a_{12} < 0$ or $a_{21} < 0$, (ii) $a_{12}a_{21} \neq 0$, and (iii) $(a_{21} + w_1 + w_2 + a_{12}) < 0$. •

Remark 6.3: *(**Geometric and other interpretations of the conditions for the existence of periodic opinions**.)* The condition given in Lemma 6.2 ensures existence of the following ellipse in the phase plane,

$$\frac{z_1^2}{r_1} + \frac{z_2^2}{r_2} = -\frac{(w_1 + w_2 + a_{12} + a_{21})}{3} =: \upsilon > 0. \quad (18)$$

From the proof of Lemma 6.2, we see that the $\operatorname{div}(\mathbf{f}(\mathbf{z}))$ is equal to zero on the ellipse and positive (negative) inside (outside) the ellipse. Thus, any periodic orbit of (17) in the phase plane should necessarily intersect the above ellipse.

The oscillatory behavior of opinions appears only when at least one of the agents has an antagonist influence on the other. This is similar to the so-called *boomerang effect* [34], where at least one of the agents has an antagonist influence on another agent. As a result, the agent on whom there is an antagonist influence shifts its opinion away from the other agent. In such situations, the opinions of the agents could converge to a disagreement equilibrium or could possibly keep on oscillating forever never converging to an equilibrium. •

Now that we have dealt with necessary conditions for a periodic solution of (17) to exist, we conclude this section by giving sufficient conditions for the agents to exhibit periodic opinion profiles. More specifically, we give sufficient conditions on the conformity weights a_{12}, a_{21} for a Hopf bifurcation to exist (proof in Appendix).

Theorem 6.4: (***Existence of Hopf bifurcation***) For the two-agent dynamics (17) with $a_{21} \in \mathbb{R}$ as the bifurcation parameter, let $\kappa_i := \left[w_i + \frac{3m_i^2}{r_i}\right] > 0, i \in \{1, 2\}$. Let $m_1 = m_2 = m \in \mathbb{R}$ and $\left[a_{12}(\kappa_2 - \kappa_1) - \kappa_1^2\right] > 0$. Then, a family of periodic orbits of (17) bifurcates from the consensus equilibrium $\mathbf{z}^* = m\mathbf{1}$ at $a_{21} = a_{21}^* := -(\kappa_1 + \kappa_2 + a_{12})$. •

VII. Simulations

In this section, we present some simulations to demonstrate our analytical results. All the simulation results were generated using MATLAB and the ODE 45 solver. Through out this section, the i^{th} element of any parameter data vector corresponds to agent i and we use $\approx$ ($=$) signs for indicating approximate (exact) values of the provided data.

In the first set of simulations, we consider a group of 6 agents forming opinions according to (2). We assume that the agents are connected via an influence network that is shown in Figure 1. The matrix of influence weights $\mathbf{A}$ is equal to the adjacency matrix associated with the graph shown in Figure 1. In Figure 2a, we illustrate the case where the opinions of all 6 agents reach a non-neutral consensus equilibrium, with consensus value equal to 40. The model parameters used to

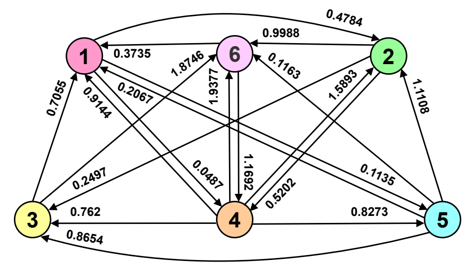


Fig. 1: A social network consisting of 6 agents. The direction of any link denotes the direction of influence and the number near arrowhead of any directed link (k, i) represents the corresponding link weight a_{ik}.

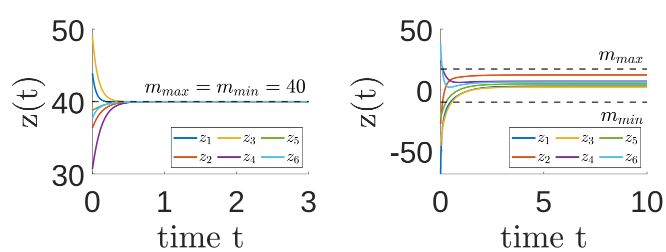
simulate this case are as follows. The vectors of initial opinions $z_i(0)$, importance weights w_i, resources r_i, agents' internal preferences p_i of all six agents are

$$\mathbf{z}_0 \approx \begin{bmatrix}43.90 & 36.34 & 49.00 & 30.69 & 38.77 & 37.63\end{bmatrix},$$
$$\mathbf{w} \approx \begin{bmatrix}1.09 & 2.98 & 2.59 & 1.82 & 1.01 & 2.65\end{bmatrix},$$
$$\mathbf{r} \approx \begin{bmatrix}317.43 & 814.54 & 789.07 & 852.26 & 505.64 & 635.66\end{bmatrix},$$
$$\mathbf{p} \approx \begin{bmatrix}225.22 & 66.40 & 71.33 & 81.16 & 165.39 & 77.99\end{bmatrix},$$

respectively. For these parameters, $m_i = 40$, $\forall i \in \mathcal{V}$, which is equal to the consensus value. Thus, this simulation verifies Theorem 5.1. For each agent $i \in \mathcal{V}$, let z_i^∞ denote its asymptotic opinion value. The vector whose each element is the absolute difference between an agent's final consensus opinion and its preference opinion ($|z_i^\infty - p_i|$) and the consensus dominance weights σ_i are

$$\mathbf{d} \approx \begin{bmatrix}185.22 & 26.40 & 31.33 & 41.16 & 125.39 & 37.99\end{bmatrix}$$
$$\boldsymbol{\sigma} \approx \begin{bmatrix}345.5 & 2424.4 & 2042.5 & 1555.1 & 510.4 & 1684.8\end{bmatrix}.$$

From this data, we can verify that the dominance claim in Proposition 5.3 is satisfied in this case.



(a) (b)

Fig. 2: Convergence of opinions. (a) Consensus equilibrium. (b) Disagreement equilibrium within $\mathcal{M}^n$.

Figure 2b depicts the scenario where the opinions of the 6 agents with no antagonistic relations converge to a disagreement equilibrium in the compact set $\mathcal{M}^n$. The model parameters in this case are,

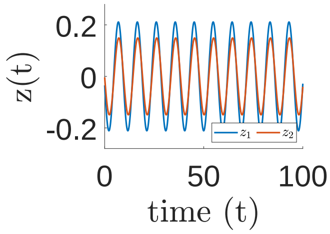
$$\mathbf{z}_0 \approx \begin{bmatrix}-69.02 & -28.03 & -46.22 & 24.01 & -23.92 & 38.38\end{bmatrix}$$
$$\mathbf{w} \approx \begin{bmatrix}0.22 & 3.48 & 0.56 & 2.56 & 0.95 & 2.62\end{bmatrix}$$
$$\mathbf{r} \approx \begin{bmatrix}317.43 & 814.54 & 789.07 & 852.26 & 505.64 & 635.66\end{bmatrix}$$
$$\mathbf{p} \approx \begin{bmatrix}-18.31 & 18.92 & -12.43 & 6.68 & 3.46 & 7.00\end{bmatrix}.$$

The agents do not achieve consensus because

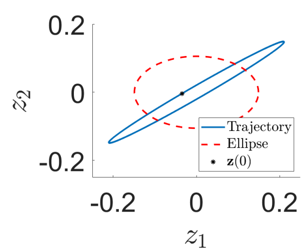
$$\mathbf{m} \approx \begin{bmatrix}-8.78 & 17.14 & -10.10 & 6.56 & 3.38 & 6.81\end{bmatrix},$$

which violates the necessary condition for consensus given in Theorem 5.1. From Figure 2b, it can be seen that every agent's opinion converges to a value in the set $\mathcal{M} \approx [-10.10, 17.14]$ which verifies the results of Theorem 4.3 and Proposition 4.5.

Figure 3a demonstrates the oscillatory behavior exhibited by opinions of two agents under (17). The model parameters for this case are, $\mathbf{z_0} \approx \begin{bmatrix}-0.0349 & -0.0039\end{bmatrix}$, $\begin{bmatrix}a_{12} & a_{21}\end{bmatrix} \approx \begin{bmatrix}-6.6667 & 3.3267\end{bmatrix}$, $\mathbf{w} \approx \begin{bmatrix}2 & 1.3333\end{bmatrix}$, $\mathbf{p} = \begin{bmatrix}0 & 0\end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix}10 & 5\end{bmatrix}$. Consider (17) with the above parameter values and let a_{21} be the bifurcation parameter. Notice that since $p_1 = p_2 = 0$, neutral consensus $\mathbf{z}^* = \mathbf{0} \in \mathcal{E}$, $\forall a_{21} \in \mathbb{R}$. These values satisfy the assumptions of Theorem 6.4 ensuring the existence a Hopf bifurcation at $a_{21}^* = 3.3333$. The eigenvalues of the Jacobian $\mathbf{J}(\mathbf{z}^*, a_{21}^*)$ are $\pm 0.6667j$. As a result, a family of periodic orbits bifurcates out of the neutral consensus equilibrium at a_{21}^* and the periodic solution shown in Figure 3a is a member of this family corresponding to $a_{21} = 3.3267 < a_{21}^*$. Figure 3b shows the corresponding trajectory in the phase plane intersecting the ellipse defined in (18). These simulations support the claim of Lemma 6.2 and its interpretation made in Remark 6.3.



(a)



(b)

Fig. 3: Oscillatory behavior of opinions. (a) Opinion trajectory. (b) Intersection of corresponding trajectory with the ellipse.

VIII. Conclusions

We proposed a non-linear model of opinion dynamics to capture the effect of heterogeneous resources available to the agents on their opinions. In contrast to our prior work [31], we dealt with general social networks with (possibly) antagonistic relations. We showed ultimate boundedness of opinions and provided sufficient conditions for the dynamics to have a globally exponentially stable equilibrium point. We also provided necessary and sufficient condition for the existence of a consensus equilibrium and quantified social dominance at consensus. Further, we showed that the set of Nash equilibria of the opinion formation game is a subset of the set of equilibrium points of the dynamics and provided conditions for these two sets to coincide. In the absence of antagonistic relations, we gave stronger results. We quantified the quality of the Nash equilibria with respect to two commonly used prices of anarchy (PoA), provided bounds on these PoA's in terms of the satisfaction ratios and proved that converging to a consensus equilibrium is a socially optimal outcome. Finally, we analyzed the periodic behavior of opinions exhibited by the proposed dynamics for the case of two agents. We provided necessary conditions for periodic solutions to exist and sufficient conditions for a Hopf bifurcation to occur at the consensus equilibrium. Future research directions include extensions of the model to a multi-topic scenario, analysis of periodic behavior exhibited by opinions in the presence of antagonistic relations for a general n-agent case, and exploration of a more general class of utility functions and resource penalty functions.

Appendix

A. Proof of Theorem 4.3

To prove this, we show that the dynamics (2) is strongly contracting. Let $\mathbf{J}(\mathbf{z}) := \left[\frac{\partial \mathbf{f}(\mathbf{z})}{\partial \mathbf{z}}\right]$ denote the Jacobian matrix of (2). The $(i, j)^{\text{th}}$ element of $\mathbf{J}(\mathbf{z})$ is

$$[\mathbf{J}(\mathbf{z})]_{ij} = \begin{cases} -w_i - \left(\displaystyle\sum_{k \in \mathcal{V} \setminus \{i\}} a_{ik} \right) - \dfrac{3z_i^2}{r_i}, & \text{if } i = j; \\ a_{ij}, & \text{if } i \neq j. \end{cases}$$

The induced $\infty-$log norm of $\mathbf{J}(\mathbf{z})$ is:

$$\mu_\infty(\mathbf{J}(\mathbf{z})) = \max_{i \in \mathcal{V}} \left[-w_i + \sum_{k \in \mathcal{N}_i^e} 2|a_{ik}| - \frac{3z_i^2}{r_i} \right].$$

Under Assumption **(A4)**, it can be seen that $\exists\, k > 0$ such that $\mu_\infty(\mathbf{J}(\mathbf{z})) < -k$; $\forall \mathbf{z} \in \mathbb{R}^n$. Now, Theorem 4.1 guarantees existence of a set $\Omega \subset \mathbb{R}^n$ which is convex, closed and positively invariant under the dynamics (2). Further, from the discussion below Theorem 4.1, we know that no equilibrium exists outside Ω. Thus, from Theorem 2.3, the dynamics (2) is strongly contracting in Ω and the unique equilibrium is globally exponentially stable. This completes the proof. ■

B. Proof of Proposition 4.5

Notice that under Assumption **(A3)**, the dynamics (2) reduces to (4). First, we prove positive invariance of $\mathcal{M}^n$ under (4) by inspecting the vector field at $\mathbf{z}$ on the boundary of $\mathcal{M}^n$. Consider any $i \in \mathcal{V}$ such that $z_i = m_{\max}$. From (7), we see that $S_i(z_i) \leq 0$ and $C_i^+(\mathbf{z}) \leq 0$, as $m_i \leq m_{\max}$ and $\bar{z}_i$ is a convex combination of $\{z_j\}_{j=1}^n$ and $\mathbf{z} \in \mathcal{M}^n$. Thus, $f_i(\mathbf{z}) \leq 0$. Similarly, for any $i \in \mathcal{V}$ such that $z_i = m_{\min}$, $f_i(\mathbf{z}) \geq 0$. This implies that $\mathcal{M}^n$ is positively invariant under (4).

Now, we show that $\mathbf{z}(t)$ converges to $\mathcal{M}^n$. Note that Assumption **(A3)** is a special case of Assumption **(A4)** and hence Theorem 4.3 guarantees that there is a unique globally exponentially stable equilibrium point $\mathbf{z}^* \in \mathcal{E}$. We know that under Assumption **(A3)**, $\bar{z}_i$ is a convex combination of $\{z_k\}_{k \in \mathcal{N}_i}$ and that (7) holds. So, $\forall \mathbf{z} \notin \mathcal{M}^n$, $\exists i \in \mathcal{V}$ such that $\dot{z}_i \neq 0$. Thus, the unique globally exponentially stable equilibrium point $\mathbf{z}^* \in \mathcal{M}^n$ and hence $\mathbf{z}(t)$ converges to $\mathcal{M}^n$.

Next, we prove the last claim in the result, i.e., for the special case of $m_{\min} < m_{\max}$. We have already seen that under Assumption **(A3)**, the unique globally exponentially stable equilibrium point $\mathbf{z}^* \in \mathcal{M}^n$. The following statement is a direct consequence of (7).

(S1) For $i \in \mathcal{V}$, $z_i^* = m_{\max}$ if and only if $m_i = m_{\max}$ and $z_j^* = m_{\max}$, $\forall j \in \mathcal{N}_i$.

From the necessary condition on m_i in **(S1)**, we can say that

(S2) $z_i^* < m_{\max},\ \forall i \in (\mathcal{V} \setminus \mathcal{V}_{\max})$.

A further consequence of **(S1)** is that for $i \in \mathcal{V}_{\max}$, if $\exists j \in \mathcal{N}_i \cap (\mathcal{V} \setminus \mathcal{V}_{\max})$ then $\bar{z}_i^* < m_{\max}$ and hence $z_i^* < m_{\max}$. This fact along with **(S1)** and **(S2)** imply that for $i \in \mathcal{V}_{\max}$, $z_i^* < m_{\max}$ if and only if there is a directed walk in G from $j \in (\mathcal{V} \setminus \mathcal{V}_{\max})$ to i. We can make similar observations corresponding to $m_{\min}$. From all these observations, we can finally say that

(S3) $\mathbf{z}^*$ lies in the interior of $\mathcal{M}^n$ if and only if $\forall i \in \mathcal{V}_{\max}$, $\exists$ a directed walk in G starting from $j \in \mathcal{V} \setminus \mathcal{V}_{\max}$ to i and $\forall i \in \mathcal{V}_{\min}$, $\exists$ a directed walk in G starting from $j \in \mathcal{V} \setminus \mathcal{V}_{\min}$ to i.

Convergence to $\mathcal{M}^n$ occurs in finite time if and only if the globally exponentially stable equilibrium $\mathbf{z}^*$ lies in the interior of $\mathcal{M}^n$. This completes the proof of the result. ∎

C. *Proof of Lemma 5.13*

Suppose Assumption **(A3)** holds. Then $\forall i \in \mathcal{V}$, it can be verified using Gerschgorin disc theorem that the Hessian of $\chi_i(\mathbf{z})$ is positive semidefinite, $\forall \mathbf{z} \in \mathbb{R}^n$. Hence, $\chi_i(\mathbf{z})$ is convex. Now, $m_i\mathbf{1}$ satisfies the first order necessary condition for a point to be a local minimizer of $\chi_i(\mathbf{z})$. Since $\chi_i(\mathbf{z})$ is convex, $m_i\mathbf{1}$ is also a minimizer of $\chi_i(\mathbf{z})$. ∎

D. *Proof of Theorem 5.14*

We get the lower bound on π_e and π_u from the Definition 5.11. To get an upper bound on π_e, note that

$$\min_{\mathbf{z}\in\mathbb{R}^n} \max_{i\in\mathcal{V}} \chi_i(\mathbf{z}) \geq \max_{i\in\mathcal{V}} \min_{\mathbf{z}\in\mathbb{R}^n} \chi_i(\mathbf{z}).$$

Using this inequality and Lemma 5.13, we can upper bound π_e defined in Definition 5.11 as, $\forall j \in \mathcal{V}$,

$$\pi_e \leq \frac{\max_{i\in\mathcal{V}} \chi_i(\mathbf{z}^*)}{\max_{i\in\mathcal{V}} \chi_i(m_i\mathbf{1})} \leq \max_{i\in\mathcal{V}} \left[\frac{\chi_i(\mathbf{z}^*)}{\chi_j(m_j\mathbf{1})}\right],$$

for any $j \in \mathcal{V}$. Choosing $j = i$, Definition 5.12 and Lemma 5.13 implies the bound on π_e given in (16). Now, consider π_u as in Definition 5.11. From Lemma 5.13, we have,

$$\min_{\mathbf{z}\in\mathbb{R}^n} \sum_{i\in\mathcal{V}} \chi_i(\mathbf{z}) \geq \sum_{i\in\mathcal{V}} \chi_i(m_i\mathbf{1}).$$

Using this we get, $\forall j \in \mathcal{V}$,

$$\pi_u \leq \frac{\sum_{i\in\mathcal{V}} \chi_i(\mathbf{z}^*)}{\sum_{i\in\mathcal{V}} \chi_i(m_i\mathbf{1})} = \sum_{i\in\mathcal{V}} \left[\frac{\chi_i(\mathbf{z}^*)}{\sum\limits_{j\in\mathcal{V}} \chi_j(m_j\mathbf{1})}\right] \leq \sum_{i\in\mathcal{V}} \frac{\chi_i(\mathbf{z}^*)}{\chi_j(m_j\mathbf{1})}.$$

Choosing $j = i$, Definition 5.12 and Lemma 5.13 implies the bound on π_u given in (16). ∎

E. *Proof of Lemma 6.1*

To prove this, rewrite (17) as,

$$\dot{z}_1 - S_1(z_1) + a_{12}z_1 = a_{12}z_2,\ \ \dot{z}_2 - S_2(z_2) + a_{21}z_2 = a_{21}z_1$$

Now, suppose $z_1(t)$ and $z_2(t)$ are periodic with fundamental periods $T_1 > 0$ and $T_2 > 0$ respectively. Then, $S_1(z_1(t))$ and $\dot{z}_1$ (resp. $S_2(z_2(t))$ and $\dot{z}_2$) are also periodic with fundamental period T_1 (resp. T_2). Then, we can see that $\exists\ m, n \in \mathbb{N}$ such that $T_1 = m\,T_2$ and $T_2 = n\,T_1$. Thus, $m = n = 1$. ∎

F. *Proof of Lemma 6.2*

From the discussion just above Lemma 6.2, it is necessary that at least one of a_{12} or $a_{21} < 0$ for periodic solutions to exist. Further, it is also necessary that both influence weights are non-zero. Otherwise, from Lemma 4.2, the socially closed agent's opinion would converge to its own m_i value. The opinion of other agent then would always converge to an equilibrium. Now, it remains to show that $(a_{21}+w_1+w_2+a_{12}) < 0$ is necessary for existence of periodic solutions. We prove the inverse using Bendixson's criteria. Note the divergence of the vector field (17) at $\mathbf{z} \in \mathbb{R}^2$ is,

$$\mathsf{div}(\mathbf{f}(\mathbf{z})) = -(w_1 + w_2 + a_{12} + a_{21}) - \frac{3z_1^2}{r_1} - \frac{3z_2^2}{r_2}.$$

Suppose $(a_{21} + w_1 + w_2 + a_{12}) > 0$. Then, $\mathsf{div}(\mathbf{f}(\mathbf{z})) < 0$, $\forall \mathbf{z} \in \mathbb{R}^2$. Thus, by Bendixson's criteria [37, Lemma 2.2] there are no periodic orbits lying entirely in $\mathbb{R}^2$.

Now, consider the case when $(a_{21} + w_1 + w_2 + a_{12}) = 0$. In this case, $\mathsf{div}(\mathbf{f}(\mathbf{z})) < 0$, $\forall \mathbf{z} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ and $\mathsf{div}(\mathbf{f}(\mathbf{z})) = 0$ iff $\mathbf{z} = \mathbf{0}$. Suppose there exists a periodic orbit, then in the simply connected region $\mathcal{S} \subset \mathbb{R}^2$ enclosed by the periodic orbit, $\iint_{\mathcal{S}} (\mathsf{div}(\mathbf{f}(\mathbf{z})))\,\mathsf{d}z_1\mathsf{d}z_2 = 0$. However, this cannot happen unless $\mathcal{S} = \{\mathbf{0}\}$, which cannot correspond to a periodic solution. Hence, again, periodic orbits cannot exist. ∎

G. *Proof of Theorem 6.4*

Under the assumption $m_1 = m_2 = m$, it follows from Theorem 5.1 that $\mathbf{z}^* = m\mathbf{1} \in \mathcal{E}$; $\forall a_{21} \in \mathbb{R}$. The Jacobian of (17) evaluated about $\mathbf{z}^*$ is,

$$\mathbf{J}(\mathbf{z}^*, a_{21}) = \begin{bmatrix} (-\kappa_1 - a_{12}) & a_{12} \\ a_{21} & (-\kappa_2 - a_{21}) \end{bmatrix}.$$

Under the stated assumptions, if $a_{21} = a_{21}^*$ then the eigenvalues of $\mathbf{J}(\mathbf{z}^*, a_{21}^*)$ are purely imaginary. This satisfies the first assumption of the Hopf bifurcation theorem [38, Theorem 3.4.2]. The eigenvalues $\lambda(a_{21})$ of $\mathbf{J}(\mathbf{z}^*, a_{21})$ which are purely imaginary at $a_{21} = a_{21}^*$ vary smoothly with the a_{21}. For values of a_{21} sufficently close to a_{21}^*, the real part of complex conjugate eigenvalue pair can be given as $Re(\lambda(a_{21})) = -0.5\,[\kappa_1 + \kappa_2 + a_{12} + a_{21}]$. Then, the derivative of $Re(\lambda(a_{21}))$ with respect to a_{21} evaluated at a_{21}^* is $\frac{\mathsf{d}}{\mathsf{d}a_{21}}\,[Re(\lambda(a_{21}))]\ |_{a_{21}=a_{21}^*} = -0.5 \neq 0$. Hence, the second assumption of the Hopf theorem is also satisfied. The claim now follows directly from the Hopf bifurcation theorem. ∎

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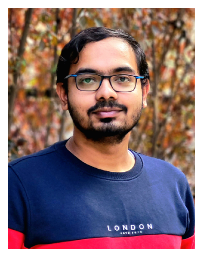

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