Event-Triggered Parameterized Control of Nonlinear Systems

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***Abstract*— This paper deals with event-triggered parameterized control (ETPC) of nonlinear systems with external disturbances. In this control method, between two successive events, each control input to the plant is a linear combination of a set of linearly independent scalar functions. At each event, the controller updates the coefficients of the parameterized control input so as to minimize the error in approximating a continuous time control signal and communicates the same to the actuator. We design an event-triggering rule (ETR) that guarantees global uniform ultimate boundedness of trajectories of the closed loop system. We also ensure the absence of Zeno behavior by showing the existence of a uniform positive lower bound on the inter-event times (IETs). We illustrate our results through numerical examples.**

***Index Terms*— Networked control systems, event-triggered control, parameterized control**

I. INTRODUCTION

EVENT-TRIGGERED control (ETC) is a commonly used control method in applications with resource constraints. Most of the ETC literature designs zero-order-hold (ZOH) sampled-data controllers. However, in many common communication protocols, including TCP and UDP [1], there is a minimum packet size. Thus, ZOH control may lead to an increase in the total number of communication instances due to under utilization of each packet. With this motivation, in this paper, we propose a non-ZOH control method and design it for control of nonlinear systems with external disturbances.

A. Literature Review

An introduction to ETC and an overview of the literature on it can be found in [2]–[5]. Typically, in ETC and in the closely related self-triggered control [6] and periodic ETC [7], control input is applied in ZOH fashion, i.e., the control input to the plant is held constant between any two successive events. There are some exceptions though. For example, in model-based ETC (MB-ETC) [8]–[12], both the controller and the actuator use identical copies of a model of the system, whose states are updated synchronously in an event-triggered manner. The model generates a time-varying control input even between two successive events. In event/self-triggered model predictive control [13]–[15], the actuator applies a part of an optimal control trajectory, which is generated by solving a finite horizon optimal control problem at each triggering instant. Recent studies in [16], [17] show that communication resources can be utilized more efficiently by transmitting only some of the samples of the generated control trajectory to the actuator, based on which a sampled data first-order-hold (FOH) control input is applied. In event-triggered dead-beat control (ET-DBC) [18], a control input sequence is transmitted to the actuator in an event-triggered manner. The actuator stores this control sequence in a buffer and applies it till the next packet is received. In team-triggered control [19], [20] each agent makes promises to its neighbors about their future states or controls and informs them if these promises are violated later.

References [21]–[23] use generalized sampled-data hold functions (GSHF) in the control of linear time-invariant systems. The idea of GSHF is to periodically sample the output of the system and generate the control by means of a hold function applied to the resulting sequence. To the best of our knowledge, this idea was first explored in the context of ETC only in our recent work [24], in which we propose an event-triggered parameterized control (ETPC) method for stabilization of linear systems. In [25], we use a similar idea to design an event-triggered polynomial controller for trajectory tracking by unicycle robots.

B. Contributions

The contributions of this paper are given below:

- We propose an ETPC method, for nonlinear systems with external disturbances, that guarantees global uniform ultimate boundedness of trajectories of the closed loop system and non-Zeno inter-event times (IETs).
- Our approach requires fewer communication packets compared to ZOH or FOH control, as our method can be fine tuned to utilize the full payload of each packet.
- Compared to MB-ETC, our method requires less computational resources at the actuator and also provides greater privacy and security.
- Compared to the ET-MPC or ET-DBC method, at each event, our proposed method requires only a limited number of parameters to be sent irrespective of the time duration of the signal.
- In this paper, we generalize the control method proposed in our previous work [24] to nonlinear control settings with external disturbances. In [24], the analysis heavily relied on linear systems theory and closed form expressions for the solutions. This approach is not applicable to nonlinear systems, wherein there are also many non-trivial technicalities that have to be taken care of. We also allow for a wider choice for the set of basis functions of the parameterized control input.
- Our recent work [25] considers a similar control method for the trajectory tracking by unicycle robots. In the current paper, we consider a more generalized problem setup and also incorporate the effect of external disturbances.

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C. Notation

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote the set of all real numbers, the set of non-negative real numbers and the set of positive real numbers, respectively. Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of all positive and non-negative integers, respectively. For any $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm. A continuous function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathscr{K}_\infty$ if it is strictly increasing, $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$. For any right continuous function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $t \geq 0$, $f(t^+) := \lim_{s \to t^+} f(s)$. For any continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$D^+ f(t) := \limsup_{h \to 0^+} \frac{f(t+h) - f(t)}{h}$$

denotes the upper right hand derivative of f. For any two functions $v, w : [0,\infty] \to \mathbb{R}_{\geq 0}$, let

$$\langle v, w \rangle_T := \int_0^T v(\tau) w(\tau) \mathrm{d}\tau.$$

Note that $\langle v, w \rangle_T$ is the inner product of the functions $v(\tau)$ and $w(\tau)$ restricted to the domain of τ to $[0,T]$.

II. PROBLEM SETUP

In this section, we present the system dynamics, the parameterized control law and the objective of this paper.

System Dynamics and Control Law

Consider a nonlinear system with external disturbance,

$$\dot{x} = f(x,u,d), \quad \forall t \geq t_0 = 0, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^q$ and $t \in \mathbb{R}_{\geq 0}$, respectively, denote the system state, the control input, the external disturbance and the time. In this paper, we consider event-triggered sampled data non-ZOH control. We call our proposed method *event-triggered parametrized control (ETPC)*.

Specifically, consider a set of functions

$$\Phi := \left\{ \phi_j : [0,\infty] \to \mathbb{R} \right\}_{j=0}^{p}.$$

We let the i^{th} control input, for $i \in \{1, 2, \ldots, m\}$, between two successive events be

$$u_i(t_k + \tau) = g(\mathbf{a}_i(k), \tau) := \sum_{j=0}^{p} a_{ji}(k) \phi_j(\tau), \ \forall \tau \in [0, t_{k+1} - t_k).$$

Here $(t_k)_{k \in \mathbb{N}_0}$ is the sequence of communication time instants, which are determined in an event-triggered manner. At t_k, the controller updates the coefficients of the parameterized control input, $\mathbf{a}(k) := [a_{ji}(k)] \in \mathbb{R}^{(p+1) \times m}$, and communicates them to the actuator. Letting $\mathbf{a}_i(k)$ denote the i^{th} column of a(k), and $\phi(\tau) := \begin{bmatrix} \phi_0(\tau) & \phi_1(\tau) & \ldots & \phi_p(\tau) \end{bmatrix}^\top$, we can write the control law as,

$$u(t_k + \tau) = \mathbf{a}^\top(k) \phi(\tau), \ \forall \tau \in [0, t_{k+1} - t_k). \tag{2}$$

The general configuration of the ETPC system is depicted in Figure 1. Here, the system state is continuously available to the controller which has enough computational resources to evaluate the event-triggering condition and to update the coefficients of the control input at an event-triggering instant.

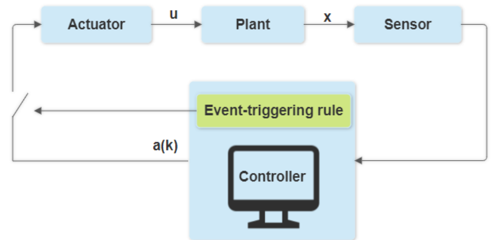


Fig. 1: Event-triggered parameterized control configuration

Assumptions

We make the following assumptions throughout this paper.

(A1) There exist $\gamma : \mathbb{R}^n \to \mathbb{R}^m$ and a continuously differentiable Lyapunov-like function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$

$$\frac{\partial V}{\partial x} f(x, \gamma(x) + e, d) \leq -\alpha_3(\|x\|) + \rho_1(\|e\|) + \rho_2(\|d\|)$$

where $\alpha_1(.)$, $\alpha_2(.)$, $\alpha_3(.)$, $\rho_1(.)$, $\rho_2(.)$ are class $\mathscr{K}_\infty$ functions and $e \in \mathbb{R}^m$, $e := u - \gamma(x)$ is the "actuation error" between the control u and $\gamma(x)$.

(A2) $f(.)$ and $\gamma(.)$ are Lipschitz on compact sets, with $f(0) = 0$ and $\gamma(0) = 0$.

(A3) There exists $D \geq 0$ such that $\|d(t)\| \leq D$, $\forall t \geq t_0$.

Note that, Assumption **(A1)** indicates that there exists a continuous-time feedback controller that makes the system (1) input-to-state-stable (ISS) with respect to the "actuation error" e and the external disturbance d. Assumption **(A2)** is a common technical assumption in the literature on nonlinear systems. Finally, Assumption **(A3)** means that the disturbance signal is uniformly upper bounded, which is again common in the literature. Throughout this paper, we make the following standing assumption regarding Φ.

(A4) Each function $\phi_j \in \Phi$ is continuously differentiable and ϕ_0 is a non-zero constant function. Let T be a fixed parameter and suppose Φ is a set of linearly independent functions when restricted to $[0,T]$, i.e., $\sum_{j=0}^{p} c_j \phi_j(t) = 0$, $\forall t \in [0,T]$ iff $c_j = 0$, $\forall j \in \{0, 1, \ldots, p\}$.

Objective

Our aim is to design a parameterized control law (2) and an event-triggering rule (ETR) for implicitly determining the communication instants $(t_k)_{k \in \mathbb{N}_0}$ so that the trajectories of the closed loop system are globally uniformly ultimately bounded while ensuring a uniform positive lower bound on the IETs.

III. DESIGN AND ANALYSIS OF EVENT-TRIGGERED CONTROLLER

In this section, we first design a parameterized control law and an ETR to achieve our objective. Then, we analyze the designed control system.

A. Control Law

The proposed control method is based on the idea of emulating a continuous time model based control signal using a parametrized time-varying signal as in (2). In particular, consider the following model for some time horizon T,

$$\dot{\hat{x}} = f(\hat{x}, \gamma(\hat{x}), 0), \ \forall t \in [t_k, t_k + T], \quad \hat{x}(t_k) = x(t_k), \ k \in \mathbb{N}_0. \quad (3)$$

Here $\hat{x}$ is the state of the model, which is the same as (1) but with $u = \gamma(\hat{x})$ and $d = 0$. The model state is reinitialized with $\hat{x}(t_k) = x(t_k)$ at each event time t_k for $k \in \mathbb{N}_0$. Now consider the open-loop control signals

$$\hat{u}_i(\tau) := \gamma_i(\hat{x}(t_k + \tau)), \quad \forall i \in \{1, 2, \dots, m\},$$

where $\gamma_i(\hat{x}(t_k + \tau))$ is the i^{th} component of $\gamma(\hat{x}(t_k + \tau))$. One way to potentially reduce the number of communication instances is to transmit the whole control signal $\hat{u}(\tau)$ for $\tau \in [0, T]$. For example, this is what is done in ET-DBC [18] and ET-MPC [13], [14]. However, transmitting the whole control signal $\gamma(\hat{x}(t_k + \tau))$ for $\tau \in [0, T)$ in a communication packet at t_k may be too costly.

So, in our proposed idea, we approximate $\hat{u}_i(\tau)$ for each i in the linear span of Φ. Specifically, we solve the following finite horizon optimization problem to determine the coefficients of the parameterized control signal (2) that is to be applied starting at t_k. For $i \in \{1, 2, \dots, m\}$,

$$\begin{aligned} \mathbf{a}_i(k) \in \arg\min_{a \in \mathbb{R}^{p+1}} & \int_0^T |g(a, \tau) - \hat{u}_i(\tau)|^2 d\tau, \\ \text{s.t.} \quad & |g(a, 0) - \hat{u}_i(0)| \leq \eta(\|\hat{x}(t_k)\|) \end{aligned} \quad (4)$$

for a function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\eta(0) = 0$ and for a finite time horizon $T > 0$. The function η and the time horizon T are to be designed. Note that, we require the signal $\hat{u}(\tau)$ for $\tau \in [0, T]$ to solve the optimization problem (4). This signal can be obtained by numerically simulating the $\hat{x}$ dynamics (3).

Remark 1. *(Control input for $\tau > T$). With the parameters $\boldsymbol{a}(k)$ obtained by solving (4), the control input applied by the actuator is as given in (2). Since t_k's are implicitly determined by an ETR online, it may happen that $t_{k+1} - t_k > T$. However, even though we find $\boldsymbol{a}(k)$ by using $\hat{u}(\tau)$ for $\tau \in [0, T]$, $g(\boldsymbol{a}_i, \tau)$, for each i, is well defined $\forall \tau \in [0, \infty]$. Hence, the control input $u(t_k + \tau)$ for τ is well defined for the entire interval $[t_k, t_{k+1})$ even if $t_{k+1} - t_k > T$.* •

Note that, we can rewrite the optimization problem (4) as,

$$\min_{a \in \mathbb{R}^{p+1}} J_i(a) := \frac{1}{2} a^\top \mathbf{H} a - b_i^\top(k) a + c_i(k), \ \text{s.t.} \ \ y^\top a - z_i(k) \leq 0, \quad (5)$$

where,

$$\mathbf{H} = 2 \begin{bmatrix} \langle \phi_0, \phi_0 \rangle_T & \langle \phi_0, \phi_1 \rangle_T & \dots & \langle \phi_0, \phi_p \rangle_T \\ \langle \phi_1, \phi_0 \rangle_T & \langle \phi_1, \phi_1 \rangle_T & \dots & \langle \phi_1, \phi_p \rangle_T \\ \dots & \dots & \dots & \dots \\ \langle \phi_p, \phi_0 \rangle_T & \langle \phi_p, \phi_1 \rangle_T & \dots & \langle \phi_p, \phi_p \rangle_T \end{bmatrix}, \quad (6)$$

$$b_i^\top(k) := 2 \begin{bmatrix} \langle \hat{u}_i, \phi_0 \rangle_T & \cdots & \langle \hat{u}_i, \phi_p \rangle_T \end{bmatrix}, \ c_i(k) := \langle \hat{u}_i, \hat{u}_i \rangle_T$$

$$y^\top := \begin{bmatrix} \phi^\top(0) \\ -\phi^\top(0) \end{bmatrix}, \quad z_i(k) = \begin{bmatrix} \hat{u}_i(0) + \eta(\|\hat{x}(t_k)\|) \\ -\hat{u}_i(0) + \eta(\|\hat{x}(t_k)\|) \end{bmatrix},$$

where recall that $\phi(\tau) = \begin{bmatrix} \phi_0(\tau) & \phi_1(\tau) & \dots & \phi_p(\tau) \end{bmatrix}^\top$.

Proposition 2. *Problem (5) is a strictly convex optimization problem and it is always feasible. Problem (5) always has exactly one optimal solution.*

Proof. The Hessian of $J_i(.)$ for all $i \in \{1, 2, \dots, m\}$, is $\mathbf{H}$. Note that $\mathbf{H}$ is twice the Gram matrix for the functions in Φ. Also, Φ is a set of linearly independent functions when restricted to $[0, T]$. Thus, we can say that $\mathbf{H}$ is a positive definite matrix. Hence, the cost function in (5) is strictly convex. Note that the only constraints in the optimization problem (5) are two linear inequality constraints in a. Thus, (5) is a strictly convex optimization problem. Problem (5) is always feasible as the choice $a_0 \phi_0(0) = \hat{u}_i(0)$ and $a_i = 0$ for $i \in \{1, \dots, p\}$, which gives $g(\mathbf{a}_i(k), \tau)$ as the zero order hold signal, satisfies the constraints. The final claim is now obvious. □

Remark 3. *(Algebraic method to solve Problem (5)). While one may simply rely on standard optimization solvers to solve Problem (5), one may also choose to solve a set of algebraic equations to obtain the unique solution to it, specially if the number of functions $p + 1$ in Φ is small. Moreover, the algebraic method provides useful properties of the solutions of Problem (5) as a function of $\hat{x}(t_k) = x(t_k)$. This is necessary for the analysis of the overall ETC system.*

The Lagrangian corresponding to Problem (5) is $\mathcal{L}_i(a, \mu) = J_i(a) + \mu_i^\top (y^\top a - z_i(k))$, where $\mu_i \in \mathbb{R}^2$ is the Lagrange multiplier. Since the Problem (5) is a strictly convex quadratic program with two linear inequality constraints, strong duality holds for problem (5) and any optimal primal-dual solution $(\boldsymbol{a}_i(k), \mu_i(k))$ must satisfy the Karush-Kuhn-Tucker (KKT) conditions. The stationarity conditions can be represented as,

$$\boldsymbol{H} \boldsymbol{a}_i(k) - b_i(k) + y \mu_i(k) = 0.$$

Further, the complementary slackness conditions are

$$\mu_{ij}(k) (y_j^\top \boldsymbol{a}_i(k) - z_{ij}(k)) = 0, \quad j \in \{1, 2\},$$

where y_j denotes the j^{th} column of y for $j \in \{1, 2\}$.

Now, we have three different cases. Case 1: both the constraints are inactive. Case 2: Only the first constraint is active. Case 3: Only the second constraint is active. Note that, in problem (5), both the constraints can not be active at the same time. In Case 1, $\boldsymbol{a}_i(k) = \boldsymbol{H}^{-1} b_i(k)$ as $\mu_i(k) = 0$. In Case 2, $y_1^\top \boldsymbol{a}_i(k) = z_{i1}(k)$ and $\boldsymbol{H} \boldsymbol{a}_i(k) - b_i(k) + y_1 \mu_{i1}(k) = 0$ as $\mu_{i2}(k) = 0$. By using these facts, we can write,

$$\begin{bmatrix} \boldsymbol{a}_i(k) \\ \mu_{i1}(k) \end{bmatrix} = \begin{bmatrix} \boldsymbol{H} & y_1 \\ y_1^\top & 0 \end{bmatrix}^{-1} \begin{bmatrix} b_i(k) \\ z_{i1}(k) \end{bmatrix} =: M \begin{bmatrix} b_i(k) \\ z_{i1}(k) \end{bmatrix}.$$

Note that the matrix M depends only on the set of functions Φ and not on $\hat{x}(t_k) = x(t_k)$. We can find a similar closed form expression of $\boldsymbol{a}_i(k)$ in Case 3. In general, one can compute the candidate solutions for each of the three cases and pick the one that satisfies the corresponding constraints. •

B. Event-Triggering Rule

We consider the following ETR, which includes two conditions. The first one is a relative thresholding condition on the actuation error e and the second condition helps to guarantee global uniform ultimate boundedness of the trajectories of the closed loop system under unknown disturbances.

$$t_{k+1} = \min\{t > t_k \ : \rho_1(\|e\|) \geq \frac{\sigma}{2}\alpha_3(\|x\|) \text{ and } V(x) \geq \varepsilon\}, \quad (7)$$

where $e{=}u - \gamma(x)$, $\varepsilon := \alpha_2(\alpha_3^{-1}(\frac{2\rho_2(D)}{\sigma})) \geq 0$ and $\sigma \in (0,1)$ is a design parameter. Here $\rho_1(.)$, $\rho_2(.)$, $\alpha_2(.)$, and $\alpha_3(.)$ are the same class $\mathcal{K}_\infty$ functions given in Assumption **(A1)**. Note that, here, e denotes the error between the actual control input u and the "ideal" feedback control input $\gamma(x)$. This error is different from the approximation error $u - \hat{u}$ as the dynamics of x and $\hat{x}$ are different.

In summary, the closed loop system, $\mathcal{S}$, is the combination of the system dynamics (1), the control law (2), with coefficients chosen by solving (4), which are updated at the events determined by the ETR (7). That is,

$$\mathcal{S}: \ (1), (2), (4), (7). \quad (8)$$

Remark 4. *(Computational requirement of the controller). We suppose that the controller has enough computational resources to evaluate the ETR (7) and to solve the finite horizon optimization problem (5) at any triggering instant. Note that, ET-MPC or ET-DBC methods also have similar computational requirements at the controller.* •

C. Analysis of the event-triggered control system

Next, we show that the trajectories of the closed loop system (8) are globally uniformly ultimately bounded and the IETs have a uniform positive lower bound that depends on the initial state of the system. First, let

$$\varepsilon_k := V(x(t_k)), \ k \in \mathbb{N}_0, \quad \bar{\varepsilon} := \max\{\varepsilon, \varepsilon_0\}.$$

Following lemmas help to prove the main result of this paper.

Lemma 5. *Consider system (8) and let Assumptions **(A1)** - **(A4)** hold. Let $\eta(.) := \frac{1}{\sqrt{m}}\rho_1^{-1}(r\frac{\sigma}{2}\alpha_3(.))$ with $r \in [0,1)$. Then, $V(x(t)) \leq \varepsilon_k \leq \bar{\varepsilon}, \forall t \in [t_k, t_{k+1})$ and $\forall k \in \mathbb{N}$.*

Proof. Let us first calculate the time derivative of $V(.)$ along the trajectories of system (8) as follows,

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x} f(x,u,d) = \frac{\partial V}{\partial x} f(x, \gamma(x) + e, d),\\ &\leq -\alpha_3(\|x\|) + \rho_1(\|e\|) + \rho_2(D),\\ &\leq -(1-\sigma)\alpha_3(\|x\|) - \left(\frac{\sigma}{2}\alpha_3(\|x\|) - \rho_1(\|e\|)\right), \forall V(x) \geq \varepsilon.\end{aligned}$$

The first inequality follows from Assumptions **(A1)** and **(A3)**. The last inequality follows from the fact that $V(x) \geq \varepsilon$ implies $\|x\| \geq \alpha_2^{-1}(\varepsilon) = \alpha_3^{-1}\left(\frac{2\rho_2(D)}{\sigma}\right)$. Note that e may be discontinuous at t_k as the control input u is updated at t_k and according to the inequality constraint in (4), $\|e(t_k^+)\| \leq \sqrt{m}\eta(\|x(t_k^+)\|)$, $\forall k \in \mathbb{N}_0$. From the definition of $\eta(.)$ and the fact that $r \in [0,1)$, we see that $\rho_1(\|e(t_k^+)\|) < \frac{\sigma}{2}\alpha_3(\|x(t_k^+)\|)$, $\forall k \in \mathbb{N}_0$. Further, the ETR (7) implies that $V(x(t_k)) = \varepsilon_k \geq \varepsilon$, $\forall k \in \mathbb{N}$ and as $x(t)$ is continuous for all t, $\dot{V}(x(t_k^+)) \leq -(1-\sigma)\alpha_3(\|x(t_k^+)\|) < 0$, $\forall k \in \mathbb{N}$.

Now, let us prove the statement that $V(x(t)) \leq \varepsilon_k$, $\forall t \in [t_k, t_{k+1})$ and $\forall k \in \mathbb{N}$ by contradiction. Suppose that this statement is not true. Then, as $V(x(t))$ is a continuous function of time, there must exist $\bar{t} \in (t_k, t_{k+1})$, for some $k \in \mathbb{N}$, such that $V(x(\bar{t})) = \varepsilon_k$ and $\dot{V}(x(\bar{t})) > 0$. However, since $\varepsilon_k \geq \varepsilon$ and the ETR is not satisfied at $t = \bar{t}$, $\rho_1(\|e(\bar{t})\|) < \frac{\sigma}{2}\alpha_3(\|x(\bar{t})\|)$, which means that $\dot{V}(x(\bar{t})) \leq -(1-\sigma)\alpha_3(\|x(\bar{t})\|) < 0$. As there is a contradiction, we conclude that there does not exist such a $\bar{t}$ and hence $V(x(t)) \leq \varepsilon_k$, $\forall t \in [t_k, t_{k+1})$ and $\forall k \in \mathbb{N}$.

Finally, if $\varepsilon_0 \leq \varepsilon$ then according to the ETR (7), $\varepsilon_1 = \varepsilon = \bar{\varepsilon}$. If $\varepsilon_0 > \varepsilon$ then following similar arguments as before we can show that $V(x(t)) \leq \varepsilon_0$, $\forall t \in [t_0, t_1)$ and hence $\varepsilon_1 \leq \varepsilon_0 = \bar{\varepsilon}$. Thus, the claim that $\varepsilon_k \leq \bar{\varepsilon}$, $\forall k \in \mathbb{N}$ follows by induction. □

Note that Lemma 5 does not impose any restrictions on $x(t_0)$. In particular, it is possible that $V(x(t_0)) < \varepsilon$. Lemma 5 only makes a claim about $V(x(t))$, $\forall t \in [t_k, t_{k+1})$, $\forall k \in \mathbb{N}$.

Lemma 6. *Consider system (8) and let Assumptions **(A1)** - **(A4)** hold. Let $\eta(.) := \frac{1}{\sqrt{m}}\rho_1^{-1}(r\frac{\sigma}{2}\alpha_3(.))$ with $r \in [0,1)$. Then, there exist $\beta_1, \beta_2 > 0$ such that $\|u(t)\| \leq \beta_1$ and $\|\dot{u}(t)\| \leq \beta_2$, $\forall t \in [t_k, \min\{t_{k+1}, t_k + T\})$, $\forall k \in \mathbb{N}_0$.*

Proof. Recall that, $\forall i \in \{1, 2, \ldots, m\}$ and $\forall k \in \mathbb{N}_0$, $u_i(t)$ for $t \in [t_k, t_{k+1})$ is chosen by solving the problem (4), which is equivalent to (5). Now, we consider the compact set $R := \{x \in \mathbb{R}^n : V(x) \leq \bar{\varepsilon}\}$. According to Lemma 5, for any $k \in \mathbb{N}_0$, $\hat{x}(t_k) = x(t_k) \in R$, which implies that $\hat{x}(t) \in R$ for all $t \in [t_k, t_k + T)$ as $\dot{V}(\hat{x}) \leq -\alpha_3(\hat{x}) \leq 0$. Now, note that,

$$|\hat{u}_i(\tau)| \leq \|\gamma(\hat{x}(t_k + \tau))\| \leq L_\gamma,$$

for some $L_\gamma > 0$. The last inequality follows from the fact that $\gamma(.)$ is Lipschitz on the compact set R with $\gamma(0) = 0$ and $\|\hat{x}\| \leq \alpha_1^{-1}(\bar{\varepsilon})$ for all $\hat{x} \in R$. The fact $\hat{x}(t_k) = x(t_k) \in R$, $\forall k \in \mathbb{N}_0$, further implies that $\eta(\|x(t_k)\|)$ and $|\hat{u}_i(0)|$ are upper bounded, $\forall k \in \mathbb{N}_0$ and thus, $\|b_i(k)\|$ and $\|z_i(k)\|$ are also uniformly, $\forall k \in \mathbb{N}_0$, upper bounded by some constants. This along with the algebraic solution of $\mathbf{a}_i(k)$ given in Remark 3, implies that $\|\mathbf{a}_i(k)\|$ is upper bounded by a constant for each $i \in \{1, 2, .., m\}$, $\forall k \in \mathbb{N}_0$. Since each $\phi_j(.) \in \Phi$ is continuously differentiable on $[0, T]$, we can say that there exist $\beta_1', \beta_2' > 0$ such that $\|\phi(t - t_k)\| \leq \beta_1'$ and $\left\|\frac{\mathrm{d}}{\mathrm{d}t}\phi(t - t_k)\right\| \leq \beta_2'$, $\forall t \in [t_k, \min\{t_{k+1}, t_k + T\}]$, $\forall k \in \mathbb{N}_0$. Putting it all together along with (2) proves the result. □

Now, we present the main result of this paper.

Theorem 7. *(Absence of Zeno behavior and global uniform ultimate boundedness). Consider system (8) and let Assumptions **(A1)** - **(A4)** hold. Let $\eta(.) := \frac{1}{\sqrt{m}}\rho_1^{-1}(r\frac{\sigma}{2}\alpha_3(.))$.*

- *If $r \in \left[0, \frac{\alpha_3(\alpha_2^{-1}(\varepsilon))}{\alpha_3(\alpha_1^{-1}(\bar{\varepsilon}))}\right)$, then the IETs, $t_{k+1} - t_k$, $\forall k \in \mathbb{N}$, are uniformly lower bounded by a positive number that depends on the initial state of the system.*
- *The trajectories of the closed loop system are globally uniformly ultimately bounded with global uniform ultimate bound $\alpha_1^{-1}(\varepsilon)$.*

Proof. Let us prove the first statement of this theorem. First, note that Assumption **(A1)** implies $\alpha_1(\|x\|) \le \alpha_2(\|x\|)\ \forall x \in \mathbb{R}^n$ and then by the definition of $\bar{\varepsilon}$, we can say that $r \in [0,1)$. Now, we consider the compact set $R := \{x \in \mathbb{R}^n : V(x) \le \bar{\varepsilon}\}$. Lemma 5 implies that $x(t) \in R,\ \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N}$. Note that according to the proof of Lemma 5, $\left\|e(t_k^+)\right\| \le \rho_1^{-1}(r\frac{\sigma}{2}\alpha_3(\left\|x(t_k^+)\right\|)),\ \forall k \in \mathbb{N}_0$. Now, by using the fact that $\alpha_2^{-1}(\varepsilon) \le \|x(t_k)\| \le \alpha_1^{-1}(\bar{\varepsilon}), \forall k \in \mathbb{N}$, we can say that the IET must at least be equal to the time it takes $\|e\|$ to grow from $e_1 := \rho_1^{-1}(r\frac{\sigma}{2}\alpha_3(\alpha_1^{-1}(\bar{\varepsilon})))$ to $e_2 := \rho_1^{-1}(\frac{\sigma}{2}\alpha_3(\alpha_2^{-1}(\varepsilon)))$. Note that, if r is chosen as in the statement of the result, then we can guarantee that $e_1 < e_2$. Next, we give a uniform upper-bound on $D^+\|e(t)\|,\ \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N}$. Note that $\forall i \in \{1, \ldots, m\}$,

$$
\begin{aligned}
|D^+\gamma_i(x(t))| &\le \limsup_{h \to 0^+} \frac{|\gamma_i(x(t+h)) - \gamma_i(x(t))|}{h} \\
&\le \limsup_{h \to 0^+} \frac{L\|x(t+h) - x(t)\|}{h} \\
&\le L \limsup_{h \to 0^+} \|f(x(t+h), u(t+h), d(t+h)\| \le \beta_3,
\end{aligned}
$$

where we have used the fact $x(t) \in R,\ \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N}$ from Lemma 5, and L is a Lipschitz constant for γ_i on the compact set R and the final inequality follows from the additional facts of boundedness of $\|u(t)\|$ from Lemma 6 and Assumptions **(A2)** and **(A3)**. We then have

$$
D^+\|e\| \le \|\dot{u}\| + m\beta_3 \le \beta, \quad \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N},
$$

for some $\beta > 0$. The last inequality follows from the uniform boundedness of $\|\dot{u}\|$ from Lemma 6. This implies that $t_{k+1} - t_k \ge \frac{e_2 - e_1}{\beta}$, for any $k \in \mathbb{N}$, which completes the proof of the first statement of this result.

Next, since the IETs have a uniform positive lower bound, $t_k \to \infty$ as $k \to \infty$. Thus, for all $t \ge t_0 = 0$,

$$
\dot{V} \le -(1-\sigma)\alpha_3(\|x\|) < 0, \quad \forall V(x) \ge \varepsilon,
$$

which implies the second statement of the result. □

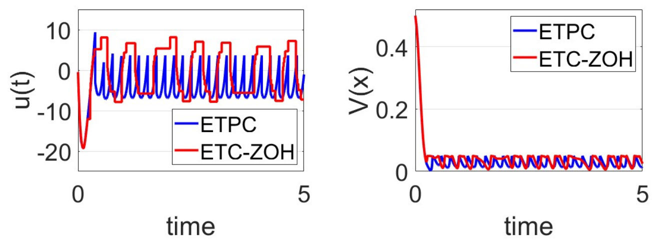
IV. NUMERICAL EXAMPLES

We illustrate our results with two numerical examples.

Example 1: (Controlled Lorenz model with disturbances)

$$
\begin{aligned}
\dot{x}_1 &= -ax_1 + ax_2 + d_1, \\
\dot{x}_2 &= bx_1 - x_2 - x_1x_3 + u + d_2, \\
\dot{x}_3 &= x_1x_2 - cx_3 + d_3,
\end{aligned} \tag{9}
$$

where $a, b, c \in \mathbb{R}_{>0}$ and $d := \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}^\top$ is the external disturbance. We set the parameter values $a = 10$, $b = 28$ and $c = 8/3$. We can show that Assumption **(A1)** and Assumption **(A2)** hold with $V(x) = \frac{1}{2}\|x\|^2$ and $\gamma(x) = -(a+b)x_1 - \frac{1}{2}x_2$. We consider the disturbance $d(t) = \frac{0.1}{\sqrt{3}}\begin{bmatrix} \sin(50t) & \sin(20t) & \sin(10t) \end{bmatrix}^\top$. Note that Assumption **(A3)** holds with $D = 0.1$. In this example, we consider the control input as a linear combination of the set of functions $\{1, \tau, \tau^2, \ldots, \tau^p\}$. Here, we compare the performance of the proposed ETPC method with the zero-order-hold control based ETC (ETC-ZOH) method. In ETC-ZOH method, we choose the same ETR (7) without further tuning of the parameters, but the control law is $u(t) = \gamma(x(t_k)),\ \forall t \in [t_k, t_{k+1})$. Figure 2 presents the simulation results with $p = 3$, $T = 0.1$ and $x(0) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top$. Figure 2a presents the evolution of control input to the plant for the proposed ETPC method and zero-order-hold ETC (ETC-ZOH). Figure 2b shows the evolution of $V(x)$ along the system trajectory for both the methods. Note that, $V(x)$ converges to the ultimate bound $\varepsilon = 0.05$ in both the cases.



(a) Control input signal (b) Evolution of $V(x(t))$

Fig. 2: Simulation results of Example 1 for $p = 3$, $T = 0.1$ and $x(0) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top$.

Next, we consider 100 initial conditions uniformly sampled from the unit sphere and we calculate the average IET (AIET) and the minimum IET (MIET) over 100 events for each initial condition with $T = 0.3$, and $p = 3$. The average of AIET over the set of initial conditions is observed as 0.0138 and 0.3649 for ETC-ZOH and ETPC, respectively. The minimum of MIET over the set of initial conditions is observed as 8.2744×10^{-4} and 0.0141 for ETC-ZOH and ETPC, respectively. Note that, for the given choice of control law and the ETR, the proposed ETPC method performs better, in terms of the AIET and MIET, compared to the ETC-ZOH method.

We repeat the procedure for different values of T and p, and the observations are tabulated in Table I. In Table I, we can see that there is a decreasing trend in the values of AIET and MIET as T increases. Note that choosing a larger T can lead to a greater fitting error in the short-term just after t_k and hence leads to smaller IETs. Whereas, there is an increasing trend in AIET and MIET as p increases. Note that choosing a larger p helps to find a better approximation of the continuous time model based control signal and hence leads to larger IETs.

TABLE I: Average of AIET and minimum of MIET, over a set of initial conditions, for different values of T and p.

	T					
	0.4		0.6		0.8	
p	**AIET**	**MIET**	**AIET**	**MIET**	**AIET**	**MIET**
3	0.2552	0.0073	0.1389	0.0036	0.0919	0.0024
4	0.3686	0.0209	0.2691	0.0068	0.1662	0.0042
5	0.3698	0.0224	0.3668	0.0166	0.3043	0.0071

Next, for the system (9) with no disturbance, we compare the performance of the ETPC method against dynamic ETC (DETC) with the following dynamic ETR proposed in [26],

$$
t_{k+1} = \min\{t > t_k\ : \nu(t) + \theta\left(\frac{\sigma}{2}\alpha_3(\|x\|) - \rho_1(\|e\|)\right) \le 0\}, \tag{10}
$$

where the dynamic variable $\nu(t)$ follows the dynamics,

$$\dot{\nu}(t) = -\omega(\nu) + \frac{\sigma}{2}\alpha_3(\|x\|) - \rho_1(\|e\|),\ \ \nu(0) = \nu_0.$$

Here, $\omega(.)$ is a class $\mathscr{K}_\infty$ function, $\nu_0 \geq 0$ and $\theta \geq 0$ are design parameters. We choose $\omega(\nu) = 0.5\nu$ and $\nu_0 = 0$. In Table II, we compare the performance of DETC method based on ZOH control, ETPC method with static ETR (7) and ETPC method with dynamic ETR (10). We can see that, for the given choice of the dynamic variable ν, the proposed ETPC method (with $T = 0.3$ and $P = 5$) performs better in terms of the AIET and MIET compared to the DETC method.

TABLE II: Average of AIET and minimum of MIET, over a set of 100 initial conditions for DETC and ETPC.

	DETC-ZOH	ETPC-static	ETPC-dynamic
AIET	0.0066	0.0534	0.2550
MIET	8.7633×10^{-4}	0.0010	0.0191

Example 2: (Forced Van der Pol oscillator with disturbances)

$$\dot{x}_1 = x_2 + d_1, \quad \dot{x}_2 = (1 - x_1^2)x_2 - x_1 + u + d_2,$$

where $d := \begin{bmatrix} d_1 & d_2 \end{bmatrix}^\top$ is the external disturbance. We can show that Assumption **(A1)** and Assumption **(A2)** hold with $V(x) = x^\top P x$ where $P = \begin{bmatrix} 4.5 & 1.5 \\ 1.5 & 3 \end{bmatrix}$ and $\gamma(x) = -x_2 - (1 - x_1^2)x_2$. We consider the disturbance $d(t) = \frac{0.1}{\sqrt{2}} \begin{bmatrix} \sin(10t) & \sin(20t) \end{bmatrix}^\top$. Note that Assumption **(A3)** holds with $D = 0.1$. In this example also, we consider the control input as a linear combination of the set of functions $\{1, \tau, \ldots, \tau^p\}$. The average of AIET over a set of 100 initial conditions is observed as 0.0938 and 1.6948 for ETC-ZOH and ETPC (with $T = 1$ and $p = 3$), respectively. The minimum of MIET over the set of initial conditions is observed as 9.93×10^{-4} and 0.0096 for ETC-ZOH and ETPC, respectively.

V. CONCLUSION

In this paper, we proposed the ETPC method for control of nonlinear systems with external disturbances. We designed a parameterized control law and an ETR that guarantee global uniform ultimate boundedness of the trajectories of the closed loop system and non-Zeno behavior of the generated IETs. We illustrated our results through numerical examples. Future work includes the generalization of this control method to distributed control setting, analytical method to determine an optimal time horizon for function fitting, systematic methods for choosing the basis functions, control under model uncertainty, quantization of the parameters, time delays, and a control Lyapunov function or MPC approach to ETPC.

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