Event-triggered Control for Nonlinear Systems with Center Manifolds

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***Abstract*—In this work, we consider the problem of event-triggered implementation of control laws designed for the local stabilization of nonlinear systems with center manifolds. We propose event-triggering conditions which are derived from a local input-to-state stability characterization of such systems. The triggering conditions ensure local ultimate boundedness of the trajectories and the existence of a uniform positive lower bound for the inter-event times. The ultimate bound can be made arbitrarily small, by allowing for smaller inter-event times. Under certain assumptions on the controller structure, local asymptotic stability of the origin is also guaranteed. Two sets of triggering conditions are proposed, one for the case where the exact center manifold is known and the other for the case where only an approximation of the center manifold is computable. Two illustrative examples representative of the two scenarios are presented and the applicability of the proposed methods is demonstrated. The second example concerns the event-triggered implementation of a position stabilizing controller for the open-loop unstable Mobile Inverted Pendulum (MIP) robot.**

***Index Terms*—Event-triggered control, Center manifold theory, Input-to-state stability, Mobile Inverted Pendulum Robot**

I. Introduction

In the recent years, various methods for resource-aware implementation of control laws have emerged, that try to utilize the resources judiciously, while guaranteeing pre-specified performance. Event-triggered control [1]–[3] is one such resource-aware technique which has gained popularity and presents an alternative to time-triggered control. In event-triggered control, the control loop is closed when certain events occur in a system and not periodically as in periodic time-triggered control.

Although event-triggered controllers have been proposed for a wide variety of settings (as surveyed in [3]), the case of nonlinear systems with center manifolds has not been looked at so far. Center manifold analysis is a crucial design and analysis tool for nonlinear systems with degenerate equilibria and is widely used in the areas of control theory, bifurcation theory and multi-scale modelling [4]. The need for research in this direction stems from the non-applicability of existing results, for systems such as the Mobile Inverted Pendulum (MIP) robot [5] and tethered satellite system [6], where controllers are designed in the presence of a center manifold. In this work, we present a solution to the problem of event-triggered control of nonlinear systems with center manifolds. This work builds our work in [7], where Lyapunov-based characterisation of local input-to-state stability (LISS) was derived for nonlinear systems with center manifolds, which is used in the present work to design event-triggering conditions. In [7], hurdles in the way of designing event-triggered controllers we identified, namely, requirement of the exact knowledge of the center manifold in checking the triggering conditions and the non-applicability of existing ISS-based results [1], [8] for a large and practical subset of nonlinear systems with center manifolds.

The main contributions of this work, with respect to the state-of-the-art, are the following: In this work, event-triggered implementation of control laws designed for local stabilization of nonlinear systems with center manifolds is investigated. The proposed methods ensure Zeno-free local ultimate boundedness of the trajectories. Under some assumptions on the controller structure, Zeno-free local asymptotic stability of the origin is ensured. The systems under consideration can be categorized into two classes, differentiated by the availability of exact and approximate knowledge of the center manifold. We propose Zeno-free triggering conditions for both the cases.

The design approach presented in this article uses the LISS characterization for nonlinear systems with center manifolds that we proposed in [7]. For a subclass of the systems under consideration, the LISS characterization meets the sufficient conditions for Zeno-free triggering in [1] and [8] and triggering conditions proposed in such works can be employed directly. For systems which do not fall in this class, which includes many practical systems such as the MIP robot, these triggering conditions cannot be used, as Zeno-free triggering is not guaranteed. For nonlinear systems with center manifolds, the focus in on local stabilization of the origin, which helps us overcome the challenges associated with the latter class in a neighbourhood of the origin. In this article, we present Zeno-free triggering conditions (inspired from [1] and [9], but with necessary modifications) for local stabilization of the latter class of nonlinear systems with center manifolds. As in [9], for a large class of systems under consideration, the triggering conditions presented in this work ensure ultimate boundedness of the trajectories, to an ultimate bound that can be made arbitrarily small. Under some assumptions on the controller structure, this article also ensures Zeno-free local asymptotic stability of the origin of the closed-loop system.

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This work was partially supported by Science and Engineering Research Board under grant CRG/2019/005743.

II. NOTATIONS AND PRELIMINARIES

We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}_+$ the set of non-negative real numbers. Given two vectors y and w, $(y;w)$ denotes the concatenation of the two vectors $[y^\top \ w^\top]^\top$. We use the notation $\mathcal{B}_r$ to denote a ball of radius r centered at the origin. We denote by $|\cdot|$, the absolute value of a real number and by $||\cdot||$, the Euclidean norm of a vector or the induced 2-norm of a matrix, depending on the argument. The $n \times n$ identity matrix is denoted by $\mathbb{I}_n$. Given a matrix $A \in \mathbb{R}^{n\times n}$, $A \succ 0$ denotes that A is a symmetric positive definite matrix. A continuous function $\alpha : [0, a) \to [0, \infty)$, with $a \in (0, \infty)$, is a class-$\mathcal{K}$ function if $\alpha(0) = 0$ and it is strictly increasing. The notations $f(x) = \mathcal{O}(||x||^p)$ and $f(x) \in \mathcal{O}(||x||^p)$, $p \in \mathbb{R}$ denote that $|f(x)| \le c_1||x||^p$ for all x such that $||x|| < \epsilon$, for some $c_1, \epsilon > 0$. The set of subgradients of a convex function V at x is called the subdifferential [10], [11] of V at x and is denoted by $\partial V(x)$. The set-valued Lie derivative [12] of a scalar convex function $V : \mathbb{R}^n \to \mathbb{R}$ with respect to a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, denoted as $\tilde{L}_f V : \mathbb{R}^n \to \mathbb{R}^n$ is defined as $\tilde{L}_f V(x) = \{\zeta^\top f(x) \mid \zeta \in \partial V(x)\}$. When $V(x)$ is differentiable at x, $\tilde{L}_f V(x) = \{\nabla V(x)\}$, the unique gradient of $V(x)$ at x.

III. PROBLEM SETUP

In this section, we introduce nonlinear systems with center manifolds and present essential preliminaries from center manifold theory. Consider the nonlinear dynamical system

$$\dot{x} = f(x, u), \ \ x(t_0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a C^2 function with $f(0, 0) = 0$. The Taylor series expansion of f about $x = 0$ and $u = 0$ yields

$$\dot{x} = Ax + Bu + \tilde{f}(x, u) \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $\tilde{f}(x, u)$ constitutes the higher-order terms and satisfies $\tilde{f}(0, 0) = 0$, $\frac{\partial \tilde{f}}{\partial x}(0, 0) = 0$ and $\frac{\partial \tilde{f}}{\partial u}(0, 0) = 0$. In this work, we focus on nonlinear systems whose linearized models have uncontrollable modes on the imaginary axis. For such systems, there exists a linear transformation $x = T(y; z), T \in \mathbb{R}^{n\times n}$ such that system (2) is transformed into

$$\begin{aligned} \dot{y} &= A_1 y + \tilde{g}_1(y, z, u) \\ \dot{z} &= A_2 z + B_2 u + \tilde{g}_2(y, z, u) \end{aligned} \tag{3}$$

where $A_1 \in \mathbb{R}^{k\times k}$, $A_2 \in \mathbb{R}^{(n-k)\times(n-k)}$, $B_2 \in \mathbb{R}^{(n-k)\times m}$, $\tilde{g}_1, \tilde{g}_2$ are the nonlinearities, the real parts of the eigenvalues of A_1 are zero and the pair (A_2, B_2) is controllable. Center manifold theory provides a model-reduction technique to determine the stability of the origin $(y, z) = 0$ of system (3), by assessing the stability of a reduced system, which governs the dynamics on the invariant center manifold.

A. Controllers for nonlinear systems with center manifolds

In the rest of the work, we use the control structure $u = K(y; z) = K_{11}z + K_{12}y + K_n(y)$, introduced in [13], for the stabilization of nonlinear systems with center manifolds. The subspace $z = 0$ of system (3) can be locally asymptotically stabilized by the term $K_{11}z, K_{11} \in \mathbb{R}^{m\times(n-k)}$, under the assumptions of stabilizability of the pair (A_2, B_2). The term $K_{12}y$ and the pseudo-control term $K_n : \mathbb{R}^k \to \mathbb{R}^m$, which is a C^1 nonlinear function of y are chosen to stabilize the dynamics on the center manifold. Denoting $A_2 + B_2K_{11}$ by A_K, we arrive at the closed-loop system

$$\begin{aligned} \dot{y} &= A_1 y + \tilde{g}_1(y, z, K(y; z)) \\ \dot{z} &= A_K z + B_2K_{12}y + B_2K_n(y) + \tilde{g}_2(y, z, K(y; z)). \end{aligned} \tag{4}$$

For the results from center manifold theory to hold, the cross-coupling linear term, $B_2K_{12}y$ between the y and z subsystems in (4) must be eliminated. The cross-coupling term is eliminated using the change of variables $v \triangleq z - Ey$, where the matrix E is found by solving the equation

$$A_K E - EA_1 + B_2K_{12} = 0. \tag{5}$$

As the sum of any eigenvalue of A_1 and any eigenvalue of A_K is non-zero, we use the result from [14] which states that there exists a unique matrix $E \in \mathbb{R}^{(n-k)\times k}$ such that (5) holds. Using this change of variables [7], we arrive at

$$\begin{aligned} \dot{y} &= A_1 y + g_1(y, v + Ey, K(y; v + Ey)) \\ \dot{v} &= A_K v + g_2(y, v + Ey, K(y; v + Ey)) \end{aligned} \tag{6}$$

where g_1 and g_2 satisfy conditions

$$g_i(0, 0, 0) = 0, \quad \frac{\partial g_i}{\partial y}, \frac{\partial g_i}{\partial v}, \frac{\partial g_i}{\partial u}\bigg|_{(0,0,0)} = 0, \text{for } i = 1, 2. \tag{7}$$

The assumption stated next is a standing assumption in this article and encodes conditions for the existence of a center manifold [15].

Assumption 1. *The functions $g_1(y, v)$ and $g_2(y, v)$ are C^2 functions and satisfy conditions in* (7)*. The eigenvalues of A_1 have zero real parts and the matrix A_K is Hurwitz.*

When system (6) satisfies Assumption 1, there exists a local k-dimensional center manifold $v = h(y)$, where the smooth function $h(y)$ is found by solving the partial differential equation

$$\begin{aligned} 0 =& A_K h(y) + g_2(y, h(y) + Ey, K(y; h(y) + Ey)) \\ &- \frac{\partial h(y)}{\partial y}(A_1 y + g_1(y, h(y) + Ey, K(y; h(y) + Ey))). \end{aligned} \tag{8}$$

The dynamics on the center manifold is governed by

$$\dot{y} = A_1 y + g_1(y, h(y) + Ey, K(y; h(y) + Ey)) \tag{9}$$

which is referred to as the reduced system. If system (6) satisfies Assumption 1, then the Reduction Theorem [15] guarantees that if the origin $y = 0$ of the reduced system (9) is locally asymptotically stable (unstable), then the origin of the full system (6) is locally asymptotically stable (unstable).

B. Event-triggered control

In event-triggered implementation, the control is updated at discrete instants t_i, $i = 0, 1, 2, \ldots$, called the event times. Between two events, in the interval $[t_i, t_{i+1})$, the control is held constant to $u = K(y(t_i); v(t_i) + Ey(t_i))$. With the measurement error $e_y(t) \triangleq y(t_i) - y(t)$ and $e_v(t) \triangleq v(t_i) - v(t)$, the control can be rewritten as $u = K(y + e_y; v + e_v + E(y + e_y))$. For further analysis of the system, we introduce the transformation $w = v - h(y)$. The trajectories of system (6) tend to the center manifold ($v = h(y)$) asymptotically. This qualitative nature of the trajectories is captured by the variable w, with $w = 0$ implying the system is on the center manifold. With control $u = K(y + e_y, w + h(y) + e_v + E(y + e_y))$ and $K_1 = [K_{12} + K_{11}E \;\; K_{11}]$ and $e = (e_y, e_v)$, the dynamics (6) in the $(y; w)$ coordinates is

$$\begin{aligned} \dot{y} = f_y(y, w, e) \triangleq& A_1 y + g_1(y, h(y) + Ey, \\ & K(y; h(y) + Ey)) + N_1(y, w, e) \quad (10) \\ \dot{w} = f_w(y, w, e) \triangleq& A_K w + B_2 K_1 e + N_2(y, w, e) \end{aligned}$$

where the functions N_1 and N_2 are such that $N_i(y, 0, 0) = 0$, $\frac{\partial N_i}{\partial w}, \frac{\partial N_i}{\partial e}\big|_{(0,0,0)} = 0$, and therefore there exists a constant $\delta_{yw} > 0$ such that, in the set

$$S = \{(y; w) \mid ||(y; w)|| < \delta_{yw}\} \quad (11)$$

we have for $i = 1, 2$,

$$||N_i|| \le k_i ||(w; e)|| \le k_i(||w|| + ||e||) \quad (12)$$

where the constants $k_i > 0$ can be made arbitrarily small by decreasing δ_{yw} [15].

C. LISS of nonlinear systems with center manifolds

In this work, the design of event-triggering conditions uses the ISS based approach proposed in [1] and generalized in [8].

Definition 1 (Local input-to-state stability [16]). *The system $\dot{x} = f(x, d)$, $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$ with f being locally Lipschitz and $f(0, 0) = 0$, is said to be locally input-to-state stable in the domain $D_x \subset \mathbb{R}^n$ with respect to input d in the domain $D_d \subset \mathbb{R}^m$, if there exists a Lipschitz continuous function $V : D_x \to \mathbb{R}_+$ and class-$\mathcal{K}$ functions $\alpha, \alpha_1, \alpha_2$ and β such that $\alpha_1(||x||) \le V(x) \le \alpha_2(||x||)$ and $\zeta^\top f(x, d) \le -\alpha(||x||) + \beta(||d||)$ hold for all $x \in D_x$, $d \in D_d$ and $\zeta \in \partial V(x)$.*

The function V satisfying the above conditions is called an LISS Lyapunov function. Motivated by the unavailability of explicit characterization of input-to-state stability for nonlinear systems with center manifolds, in terms of the class-$\mathcal{K}$ functions α and β in Definition 1, we investigated this scenario in our recent work [7]. The proposition presented next establishes that a controller that locally asymptotically stabilizes system (6), renders (10) LISS with respect to measurement errors.

Proposition 1 ([7]). *Under Assumption 1, if the origin $y = 0$ of the reduced system* (9) *is locally asymptotically stable, then the overall system* (10) *is locally input-to-state stable with respect to the error $e = (e_y; e_v)$.*

Proposition 1 generalizes the *Reduction Theorem*, as in the absence of the error e, local asymptotic stability of the overall system is recovered. Note that the stability properties of system (10) are the same as that of system (1) in view of the sequence of smooth transformations $w = v - h(y)$, $v = z - Ey$ and $x = T(y; z)$ relating the two systems. As part of the proof of Proposition 1, the following explicit LISS characterization was derived in terms of functions α_D and β_G, which are class-$\mathcal{K}$ functions of $||(y; w)||$ and $||e||$ respectively.

$$\begin{aligned} \tilde{L}_{(f_y; f_w)} V \le& \underbrace{-\alpha_4(||y||) - (1 - s_f)\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||w||}_{-\alpha_D(||(y;w)||)} \\ &+ \underbrace{\left(k_v k_1 + k_2 \frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} + \frac{||PB_2 K_1||}{\sqrt{\lambda_{min}(P)}}\right)||e||}_{\beta_G(||e||)} \end{aligned} \quad (13)$$

where α_4 is a class-$\mathcal{K}$ function, $\tilde{L}_{(f_y; f_w)} V$ is the set valued Lie derivative of V with respect to the vector field $(f_y; f_w)$ as defined in (10), k_1, k_2 are constants from (12) and $s_f \in (0, 1)$.

IV. Event-triggered control

In this section, we use the LISS characterization of nonlinear systems with center manifolds to propose event-triggered control implementations. In [1] and [8], a relative threshold based event-triggered control was proposed, where the events are triggered when

$$\beta_G(||e||) \ge \sigma \alpha_D(||(y; w)||), \;\; \sigma \in (0, 1) \quad (14)$$

is satisfied at event times $t_i, i = 0, 1, 2, \ldots$. In event-triggered implementation, for $t > t_i$, the input $u(t)$ evolves as

$$u(t) = \begin{cases} K(y(t_i); z(t_i)) & \text{if } \beta_G(||e||) < \sigma \alpha_D(||(y; w)||) \\ K(y(t); z(t)) & \text{if } \beta_G(||e||) \ge \sigma \alpha_D(||(y; w)||). \end{cases} \quad (15)$$

From (13) and (15), we have $\tilde{L}_{(f_y; f_w)} V \le -(1 - \sigma)\alpha_D(||(y; w)||) < 0, \;\; \forall \; (y; w) \ne 0$ and local asymptotic stability of the origin of system (10) is guaranteed.

Two major hurdles can be seen in the checking of the triggering condition (14). 1) The triggering rule (14) can be accurately checked only when $w = v - h(y)$ is exactly computable. The center manifold $h(y)$ is found by solving (8) and there are systems for which $h(y)$ is exactly computable (one such system is presented in Example 1). However, for most systems, only an approximation of $h(y)$ can be found. 2) Sufficient conditions that rule-out Zeno behaviour by showing that the inter-event times $t_{i+1} - t_i$ are lower bounded by a positive constant for all $i \ge 0$, have been proposed in [1], [8] and these conditions require that the comparison functions α_D and β_G in (13) are such that $\alpha_D^{-1} \circ \beta_G$ is Lipschitz continuous over compact sets. This assumption on the comparison functions has since been made in [17], [18], among many others. In our case, this assumption holds only when $\alpha_4 \in \mathcal{O}(||y||^p)$, $p \le 1$. When $\alpha_4 \in \mathcal{O}(||y||^p)$, $p > 1$, the sufficient conditions from [1], [8] are not satisfied and thus no conclusion can be drawn regarding the existence or non-existence of Zeno behaviour under the implementation (15). In this article, we propose Zeno-free triggering conditions for this class of systems.

A. Triggering rule guaranteeing Zeno-free asymptotic stability of the origin of the closed-loop system

In this subsection, we overcome the identified hurdles and begin with systems for which the center manifold can be exactly computed and try to overcome the second difficulty by proposing triggering conditions which are different from (14). The class of systems for which, $\alpha_4 \in \mathcal{O}(||y||^p)$, $p > 1$ in (13) arises when the function g_1 in the reduced system (9) has a polynomial approximation in a neighbourhood of the origin, that is, $||g_1|| \le k_5||y||^p$, $p > 1$ in a neighbourhood of the origin. For simplicity, in the rest of this work, we use g_1 to denote $g_1(y, h(y) + Ey, K(y; h(y) + Ey))$.

Assumption 2. ***a)*** *For the system* (6)*, the matrix* $A_1 = 0$*. In the dynamics of the reduced system* (9)*, the function* $g_1(y, v)$ *is such that* $||g_1(y, v)|| \le k_5||y||^p$ *and* $y^\top g_1(y, v) \le -k_6||y||^{p+1}$*, with* $p > 1$ *for some* $k_5, k_6 > 0$ *in a neighbourhood of the origin. The origin* $y = 0$ *of the reduced system* (9) *is locally asymptotically stable.* ***b)*** *The controller* $u = K(y; z) = K_{11}z + K_{12}y + K_n(y) = K_1(y; v) + K_n(y)$ *considered in subsection III-A is such that* $K_n(y) = 0$ *and the matrix* $K_1 \triangleq [K_{12} + EK_{11} \;\; K_{11}] = [0 \;\; K_{11}]$*.*

Models and controllers of Example 1 and the Mobile Inverted Pendulum robot considered in this work satisfy the conditions of Assumption 2.

Lemma 1 ([10, page 181])**.** *Let* $P \in \mathbb{R}^{n_1\times n_1}$ *be a symmetric positive definite matrix and* $P = M^\top DM$ *be its eigen-decomposition, where* $M \in \mathbb{R}^{n_1\times n_1}$ *is an orthonormal matrix and* $D \in \mathbb{R}^{n_1\times n_1}$ *is a diagonal matrix. The subdifferential of the convex function* $f_P = \sqrt{x^\top Px} : \mathbb{R}^{n_1} \to \mathbb{R}_+$ *at* $x = 0$ *is* $\partial f_P(0) = \{\zeta \in \mathbb{R}^{n_1} : ||\zeta^\top MD^{-\frac{1}{2}}|| \le 1\}$*.*

Proposition 2. *Consider the system* (6)*. If the conditions in Assumptions 1 and 2 are satisfied, then the origin of the overall system* (10) *is locally asymptotically stable under the event-triggering condition*

$$\begin{aligned} &||e_v|| \ge \sigma(||w|| + ||y||^{(p+1)}) \\ &0 < \sigma \le \frac{(1 - s_f)\lambda_{min}(Q)}{2||PBK_1||}\frac{\sqrt{\lambda_{min}(P)}}{\sqrt{\lambda_{max}(P)}} \end{aligned} \tag{16}$$

where $s_f \in (0, 1)$*. Moreover, the inter-event times* $t_{i+1} - t_i$ *are lower bounded by a positive constant for all* $i \ge 0$*.*

Proof. Consider the LISS Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}_+$

$$V = ||y|| + \sqrt{w^\top Pw}. \tag{17}$$

The function V is continuously differentiable everywhere except on the set $N_e = \{(y; w) \in \mathbb{R}^n : w = 0 \text{ or } y = 0\}$. The functions $\alpha_1(||(y; w)||) = \min\left\{1, \sqrt{\lambda_{\min}(P)}\right\} ||(y; w)||$ and $\alpha_2(||(y; w)||) = \sqrt{2}\max\left\{1, \sqrt{\lambda_{\max}(P)}\right\} ||(y; w)||$ are class-$\mathcal{K}$ functions satisfying $\alpha_1(||(y; w)||) \le V((y; w)) \le \alpha_2(||(y; w)||)$. Taking the time derivative of V along the trajectories of the system (10) on the set $\mathbb{R}^n \setminus N_e$, we obtain

$$\dot{V} = \frac{y^\top \dot{y}}{||y||} + \frac{1}{2\sqrt{w^\top Pw}}\left(\dot{w}^\top Pw + w^\top P\dot{w}\right).$$

By Assumption 2, $A_1 = 0$, the function g_1 is such that $y^\top g_1 \le -k_6||y||^{p+1}$, the control $u = K_{11}(v+e_v)$, $B_2K_1e = B_2K_{11}e_v$ in (10) and the functions N_1 and N_2 are functions of y, w and e_v. This leads us to

$$\begin{aligned} \dot{V} \le& -k_6||y||^p + \frac{y^\top N_1(y, w, e_v)}{||y||} \\ &+ \frac{1}{2\sqrt{w^\top Pw}}\left((A_Kw + B_2K_{11}e_v + N_2(y, w, e_v))^\top Pw\right. \\ &\left.+w^\top P(A_Kw + B_2K_{11}e_v + N_2(y, w, e_v))\right) \\ \le& -k_6||y||^p + ||N_1(y, w, e_v)|| - \frac{w^\top Qw}{2\sqrt{w^\top Pw}} \\ &+ \frac{1}{\sqrt{w^\top Pw}}(w^\top PB_2K_{11}e_v + w^\top PN_2(y, w, e_v)) \\ \le& -k_6||y||^p - \frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||w|| + k_1(||w|| + ||e_v||) \\ &+ \frac{||PB_2K_{11}||}{\sqrt{\lambda_{min}(P)}}||e_v|| + \frac{k_2\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}}(||e_v|| + ||w||). \end{aligned}$$

With $s_f \in (0, 1)$,

$$\begin{aligned} \dot{V} \le& -k_6||y||^p - (1 - s_f)\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||w|| \\ &+ \left(k_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} - s_f\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}\right)||w|| \\ &+ \left(k_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} + \frac{||PB_2K_{11}||}{\sqrt{\lambda_{min}(P)}}\right)||e_v||. \end{aligned} \tag{18}$$

Using the notation $m_{p_2} \triangleq \frac{||PB_2K_{11}||}{\sqrt{\lambda_{min}(P)}}$ and $\bar{m}_{p_2} \triangleq \left(k_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} + \frac{||PB_2K_{11}||}{\sqrt{\lambda_{min}(P)}}\right)$ and with the event-triggering rule ensuring $||e_v|| \le \sigma(||w|| + ||y||^{p+1})$ between any two events (that is $\forall\, t \in [t_i, t_{i+1})$), we have

$$\begin{aligned} \dot{V} \le& -k_6||y||^p + \bar{m}_{p_2}\sigma||y||^{p+1} - (1 - s_f)\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||w|| \\ &+ m_{p_2}\sigma||w|| + \left((1 + \sigma)\left(k_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}}\right)\right. \\ &\left.-s_f\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}\right)||w||. \end{aligned}$$

With $s_y \in (0, 1)$,

$$\begin{aligned} \dot{V} \le& -k_6(1 - s_y)||y||^p + (\bar{m}_{p_2}\sigma||y||^{p+1} - k_6s_y||y||^p) \\ &- (1 - s_f)\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||w|| + m_{p_2}\sigma||w|| \\ &+ \left((1 + \sigma)\left(k_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}}\right)\right. \\ &\left.-s_f\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}\right)||w||. \end{aligned} \tag{19}$$

With σ chosen as in (16), in a small neighbourhood S in (11), the constants k_1, k_2 from (12) can be chosen such that the last term in Equation (19) is less than or equal to zero and $(\bar{m}_{p_2}\sigma||y||^{p+1} - k_6 s_y||y||^p) \leq 0$. For $(y;w) \in S$, we have

$$\begin{aligned}\dot{V} &\leq -k_6(1-s_y)||y||^p - (1-s_f)\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||w|| \\ &\triangleq -w_s((y;w)) < 0,\end{aligned} \tag{20}$$

where $w_s((y;w))$ is a positive definite function of $(y;w)$. It can be verified that, on the set N_e where the Lyapunov function (17) is non-differentiable, the inequality (20) holds, that is, $\zeta^\top(f_y;f_w) \leq -w_s((y;w))$, for all $\zeta \in \partial V$ and ∂V is found through Lemma 1. Therefore, $\tilde{L}_{(f_y;f_w)}V \leq -w_s((y;w))$, $\forall\, t \in [t_i, t_{i+1})$.

Next, we prove the existence of a uniform positive lower bound for the inter-event times $t_{i+1} - t_i$, when the system is initialized in the positively invariant set S_v. The error $e_v(t)$ is defined to be $v(t_i) - v(t) = (w(t_i) + h(y(t_i))) - (w(t) + h(y(t))) = e_w(t) + e_h(t)$. Consider

$$\begin{aligned}&\frac{d}{dt}\left(\frac{||e_v||}{||w|| + ||y||^{p+1}}\right) \\ &= \frac{e_v^\top \dot{e}_v}{(||w|| + ||y||^{p+1})\,||e_v||} - \frac{\left(\frac{w^\top \dot{w}}{||w||} + \frac{(p+1)||y||^p y^\top \dot{y}}{||y||}\right)||e_v||}{(||w|| + ||y||^{p+1})^2} \\ &\leq \frac{||\dot{e}_v||}{(||w|| + ||y||^{p+1})} + \frac{(||\dot{w}|| + (p+1)||y||^p||\dot{y}||)\,||e_v||}{(||w|| + ||y||^{p+1})^2} \\ &\leq \frac{||\dot{e}_w|| + ||\dot{e}_h||}{(||w|| + ||y||^{p+1})} + \frac{(||\dot{w}|| + (p+1)||y||^p||\dot{y}||)\,||e_v||}{(||w|| + ||y||^{p+1})^2}.\end{aligned} \tag{21}$$

From Assumption 2, $||g_1|| \leq k_5||y||^p$ for some $k_5 > 0$. As $||h(y)|| \in \mathcal{O}(||y||^2)$ and $||\frac{\partial h(y)}{\partial y}|| \in \mathcal{O}(||y||)$, there exist constants $k_7 > 0$, $k_8 > 0$ such that $||h(y)|| \leq k_7||y||^2$ and $||\frac{\partial h(y)}{\partial y}|| \leq k_8||y||$ in a small neighbourhood of $y = 0$. From equations (10) and (12), we have $||\dot{y}|| \leq k_5||y||^p + k_1||w|| + k_1||e_v||$, $||\dot{w}|| \leq (||A_c|| + k_2)||w|| + (||B_2K_{11}|| + k_2)||e_v||$ and $||\dot{e}_h|| = ||\dot{h}(y)|| \leq ||\frac{\partial h}{\partial y}||\,||\dot{y}|| \leq k_8k_5||y||^{p+1} + k_8k_1\delta_{yw}||w|| + \delta_{yw}k_1k_8||e_v||$ ($||y|| \leq \delta_{yw}$ has been used as $(y;w) \in S$). Consider the numerator of the first term in the right-hand side of (21). Using the inequalities derived so far, we have

$$||\dot{e}_w|| + ||\dot{e}_h|| \leq a_1\left(||w|| + ||y||^{p+1}\right) + a_2||e_v|| \tag{22}$$

where, $a_1 \triangleq \max\{||A_c|| + k_2 + k_8k_1\delta_{yw}, k_8k_5\}$ and $a_2 \triangleq ||B_2K_{11}|| + k_2 + \delta_{yw}k_1k_8$. For the numerator of the second term in (21), we have

$$\begin{aligned}&||\dot{w}|| + (p+1)||y||^p||\dot{y}|| \\ &\qquad\leq a_3\left(||w|| + ||y||^{p+1}\right)||e|| + a_4||e_v||^2\end{aligned} \tag{23}$$

where $a_3 \triangleq \max\{||A_c|| + k_2 + (p+1)\delta_{yw}^p k_1, \delta_{yw}^{(p-1)}k_5\}$ and $a_4 \triangleq ||B_2K_{11}|| + k_2 + (p+1)\delta_{yw}^p k_1$. From (22) and (23), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{||e_v||}{||w|| + ||y||^{p+1}}\right) \\ &\leq a_1 + \left(\frac{(a_2+a_3)||e_v||}{||w|| + ||y||^{p+1}}\right) + a_4\left(\frac{||e_v||}{||w|| + ||y||^{p+1}}\right)^2.\end{aligned}$$

Denoting $||e_v||/(||w|| + ||y||^{p+1})$ by e_s, we have $\dot{e}_s \leq a_1 + (a_2+a_3)e_s + a_4e_s^2$. Using the Comparison lemma [15], it follows that $e_s(t) \leq \phi(t)$, where $\phi(t)$ is the solution of $\dot{\phi} = a_1 + (a_2+a_3)\phi + a_4\phi^2$, initialized at $\phi(0) = 0$. When an event occurs (when e_s rises from zero to meet σ), the control is updated and $e_s(t)$ is reset to zero. Let τ_1 be the time taken by $\phi(t)$ to evolve from 0 to σ. As $e_s(t) \leq \phi(t)$, the time taken by $e_s(t)$ to reach σ is greater than τ_1. By the Comparison lemma,

$$e_s(t) \leq \phi(t) = b\tan\left(\frac{b}{2}(t+c)\right) - (a_2+a_3)/(2a_4).$$

where $b = \sqrt{4a_1a_4 - (a_2+a_3)^2}$ and $c = (2/b)\tan^{-1}((a_2+a_3)/b)$. $\phi(\tau_1) = \sigma$ implies

$$\begin{aligned}\tau_1 =& \frac{2}{b}\left(\tan^{-1}\left(\frac{2a_4\sigma + (a_2+a_3)}{b}\right) - \tan^{-1}\left(\frac{a_2+a_3}{b}\right)\right) \\ &> 0.\end{aligned} \tag{24}$$

Thus, for all initializations $(y(0);w(0)) \in S_v$, there exists a uniform positive lower bound for the inter-event times. Moreover, by (20), local asymptotic stability of the origin is guaranteed for all $x(0) \in S_v$, where

$$S_v = \{(y;w) \in S \mid V((y;w)) \leq \alpha_1(\delta_{yw})\} \tag{25}$$

is the largest, connected sub-level set of V contained in S. ■

B. Triggering rule guaranteeing Zeno-free local ultimate boundedness of the trajectories of the closed-loop system

In Assumption 2, the restriction on the matrix K_1 was needed to ensure both asymptotic stability and non-existence of Zeno behaviour. We now relax this assumption on the controller structure and show that under the implementation

$$\begin{aligned}u(t) &= \begin{cases}0 & \text{if } t_1 \neq t_0, \ \forall\, t \in [t_0, t_1) \\ K(y(t_i); z(t_i)) & \text{if } t \in [t_i, t_{i+1}), i \geq 1\end{cases} \\ t_1 &= \min\{t \geq t_0 \mid (y(t);w(t)) \in S_v \setminus S_2\} \\ t_{i+1} &= \min\{t \geq t_i \mid ||e|| \geq \sigma(||w|| + ||y||^{(p+1)}) \\ &\qquad\qquad \text{and } (y(t);w(t)) \in S_v \setminus S_2\}, i \geq 1\end{aligned} \tag{26}$$

where the set S_v is as defined in Equation (25) and

$$S_2 = \{(y;w) \mid ||(y;w)|| < \alpha_2^{-1} \circ \alpha_1(r_s) = r_1\}, \tag{27}$$

the trajectories of the closed-loop system are ultimately bounded by a ball of radius r_s. If $(y(0);w(0)) \in S_v \setminus S_2$, the first event instant $t_1 = t_0$ and the second case defining $u(t)$ in Equation (26) is active for all $t \geq t_0$. If $(y(0);w(0)) \in S_2$, then $t_1 \neq t_0$ and the first case defining $u(t)$ is active for $t \in [t_0, t_1)$ before the second case takes over for all $t \geq t_1$.

Proposition 3. *Consider system* (6). *If the conditions in Assumptions 1 and 2 hold and $\mathcal{B}_{r_s} \subset S_v$, then the trajectories of the system* (10) *are locally ultimately bounded by $\mathcal{B}_{r_s}$, under the event-triggered implementation* (26)*, with σ chosen according to* (16). *Moreover, the inter-event times $t_{i+1} - t_i$ are lower bounded by a positive constant for all $i \geq 0$.*

Proof. From (20), (27) and implementation (26), we have $\tilde{L}_{(f_y;f_w)}V \leq -w_s((y;w))$, $\forall\ ||(y;w)|| \geq \alpha_2^{-1} \circ \alpha_1(r_s)$, and $\forall\, t \in [t_i, t_{i+1})$.

Next we show that the inter-event times $t_{i+1} - t_i$ are uniformly lower bounded by a positive constant for all

$(y(t_0);w(t_0)) \in S_v$. Under the implementation rule (26), events are triggered only in $S_v \setminus S_2$ and the control is not updated when the trajectory enters S_2. When the system is initialized in S_2, the control is set to zero until the system leaves S_2. For each i, $||(y(t_i);w(t_i))|| \geq r_1$ and $e(t_i) = 0$. The next event occurs at t_{i+1}, when $||e||$ rises from zero and meets $\sigma(||w|| + ||y||^{p+1})$ in $S_v \setminus S_2$. In $S_v \setminus S_2$, $\sigma_l \triangleq \sigma(r_1 + r_1^{p+1}) \leq \sigma(||w|| + ||y||^{p+1}) \leq \sigma(\delta_{yw} + \delta_{yw}^{p+1}) \triangleq \sigma_u$. Next, consider the evolution of $||e||$ along the trajectories of the event-triggered closed-loop system (6)-(26), between two consecutive event instants $\frac{d||e||}{dt} \leq ||\dot{e}|| = || - (\dot{y};\dot{w})||$. The function $f(x,u)$ in (1) is twice continuously differentiable and due to the sequence of smooth coordinate transformations relating x and $(y;w)$, there exists a constant L_1 in S_1 such that $||(\dot{y};\dot{w})|| \leq L_1||(y;w;e)|| \leq L_1||(y;w)|| + L_1||e|| \leq L_1\delta_{yw} + L_1||e||$. Using the notation $\delta \triangleq L_1\delta_{yw}$,

$$\frac{d||e||}{dt} \leq || - (\dot{y};\dot{w})|| \leq L_1\delta_{yw} + L_1||e|| = L_1||e|| + \delta.$$

Using the Comparison lemma with $e(0) = 0$, we arrive at $||e(t)|| \leq \frac{\delta}{L_1}(e^{L_1 t} - 1)$. The time taken by $\frac{\delta}{L_1}(e^{L_1 t} - 1)$ to rise from 0 to σ_l serves as a lower bound for the inter-event times. The lower bound τ_3 is found by solving

$$\frac{\delta}{L_1}(e^{L_1\tau_3} - 1) = \sigma_l \implies \tau_3 = \frac{1}{L_1}\ln\left(1 + \frac{\sigma_l L_1}{\delta}\right) > 0. \quad (28)$$

Thus, we have shown that the inter-event times are uniformly lower bounded by $\tau_3 > 0$. By [15, Theorem 4.18], we conclude that the trajectories of the closed-loop system (6)-(26) are locally ultimately bounded by $\mathcal{B}_{r_s} \subset S_v$ and this ball is reached in finite time. ■

The radius r_s of the ultimate bound is a user-specified parameter and can be made arbitrarily small, but this leads to small estimates of inter-event times, as τ_3 in (28) is a function of σ_l, which grows small as the size of S_2 decreases.

V. Event-triggered control with approximate knowledge of the center manifold

The triggering conditions in Propositions 2 and 3 can be accurately checked only when the variable $w = v - h(y)$ is exactly computed. In this section, we present triggering conditions, that do not require the exact knowledge of the center manifold. The triggering conditions are of the form $||e_v|| \geq \sigma(||w^a|| + ||y||^{(p+1)})$, which possess the same structure as the triggering conditions in Propositions 2 and 3 respectively, but with $w = v - h(y)$ replaced by the approximation $w^a = v - h^a(y)$. Here, $h^a(y)$ is a polynomial approximation of $h(y)$ of degree r, found by solving (8) and they are related by $h(y) = h^a(y) + \mathcal{O}(||y||^s)$ with $s > r$.

Proposition 4. *Consider the system* (6). *If the conditions in Assumptions 1 and 2 are satisfied, then the origin of the overall system* (10) *is locally asymptotically stable under the event-triggered implementation with relative thresholding* $||e_v|| \geq \sigma(||w^a|| + ||y||^{p+1})$*, with* σ *chosen as in* (16). *Moreover, the inter-execution times* $t_{i+1} - t_i$ *are lower bounded by a positive constant for all* $i \geq 0$.

Proposition 5. *Consider the system* (6). *If the conditions in Assumptions 1 and 2 are satisfied, then the trajectories of the system* (10) *are locally ultimately bounded by* $\mathcal{B}_{r_s} \subset S_v$ *(a sub-level set of* (17) *where* (29) *holds), under the event-triggered implementation* (26) *with relative thresholding* $||e|| \geq \sigma(||w^a|| + ||y||^{(p+1)})$ *and* σ *chosen according to* (16). *Moreover, the inter-execution times* $t_{i+1} - t_i$ *are lower bounded by a positive constant for all* $i \geq 0$.

Propositions 4-5 are analogues of Propositions 2-3 from Section IV. The proofs are omitted, as they follow along similar lines as the proofs of Propositions 2-3, but with a crucial difference described next. Consider (18) in the proof of Proposition 2. With the proposed event-triggering condition, the inequality $||e_v|| \leq \sigma(||w^a|| + ||y||^{(p+1)})$ is ensured throughout the implementation. Using the inequality $||w^a|| = ||w + (h(y) - h^a(y))|| \leq ||w|| + ||(h(y) - h^a(y))|| \leq ||w|| + \mathcal{O}(||y||^s)$, we arrive at

$$\tilde{L}_{(f_y;f_w)}V \leq -k_6(1 - s_y)||y||^p - (1 - s_f)\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||w|| + \mathcal{O}(||y||^s). \quad (29)$$

The difference between (29) and (20) in the proof of Proposition 2, is the presence of $\mathcal{O}(||y||^s)$. In a neighborhood of the origin, the sum of the first and third term, which can be any polynomial in $\mathcal{O}(||y||^s)$, is less than zero. Therefore $\dot{V} < 0$ in a neighborhood of the origin and local asymptotic stability of the origin is guaranteed. The proof of non-existence of Zeno behaviour in Propositions 4 and 5 remains the same as in the proofs of Propositions 2 and 3 respectively.

VI. Simulation results

In this section, we present two examples to demonstrate the application of the triggering conditions presented in Proposition 2 and Proposition 4 respectively.

A. Example 1

Consider the system

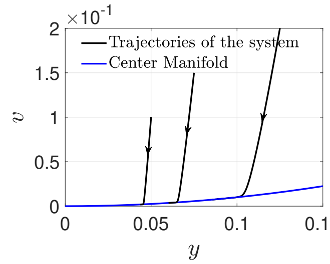
$$\dot{y} = -yv, \quad \dot{v} = v + u + y^2 - 2v^2. \quad (30)$$

The center manifold of the closed-loop system with the controller $u = -2v$, can be computed exactly [19] and is found to be $v = y^2$. The dynamics on the center manifold is $\dot{y} = g_1(y, h(y)) = -y^3$. We have $||g_1|| = |y|^3$ and $y^\top g_1 \leq -|y|^4$ with $k_5, k_6 = 1$. Therefore, Assumption 2 is satisfied. With the change of variables $w = v - y^2$, and introducing the measurement error $e_v = v(t_i) - v(t)$ as in subsection III-B, we obtain

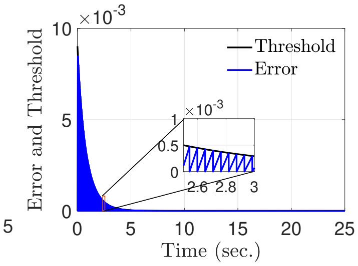
$$\dot{y} = -y^3 - yw, \quad \dot{w} = -w - 2e_v - w(w + y^2). \quad (31)$$

Now $N_1 = -yw$ and $N_2 = -w(w + y^2)$. In the set $S_1 = \{(y,w) \mid ||y|| \leq \frac{1}{\sqrt{6}} \text{ and } ||w|| \leq \frac{1}{6}\}$, we have $||N_1|| \leq \frac{1}{\sqrt{6}}||w||$ and $||N_2|| \leq \frac{1}{6}||w||$, with $k_1 = \frac{1}{\sqrt{6}}$ and $k_2 = \frac{1}{6}$. Using the triggering condition $||e_v|| \geq \sigma(||w|| + ||y||^4)$, $\sigma = 1/16$ (according to (16)) through Proposition 2, local asymptotic stability of the origin of (31) is guaranteed with Zeno-free event-triggering.

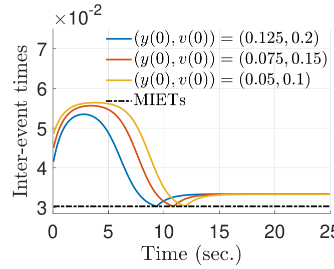
In Figure 1, simulation results of the event-triggered closed-loop system are presented. Trajectories from three initial conditions are plotted in Figure 1a along with the center manifold $v = y^2$. The trajectories tend to the center manifold quickly and the evolution along the center manifold is significantly slower in comparison. The mechanism of event-triggering is shown in Figure 1b, by plotting the evolution of the error $||e_v||$ and the threshold $\frac{1}{16}(||w|| + ||y||^4)$. The evolution of inter-event times for three initial conditions are shown in Figure 1c. The inter-event times are lower bounded by 30.3 ms. The estimates of minimum inter-event times from (24) and (28) are conservative, as they depend on Lipschitz constants and bounds on polynomial functions. To get a better estimate, the event-triggered closed-loop system was simulated for ten initial conditions $\left(0.1\cos(\frac{2\pi k_t}{10}), 0.1\sin(\frac{2\pi k_t}{10})\right)$, $k_t = 0, 1, 2, \ldots, 9$ for 25 s. The minimum inter-event time in these simulations (MIETs) was found to be 30.3 ms. To assess the performance of event-triggered control with respect to time-triggered control, we choose MIETs as the sampling time for time-triggered control. From Figure 1d, we see that the performances of time-triggered and event-triggered control are a close match. However, the number of control updates is much higher in time-triggered control, thus making a case for the use of event-triggered control.



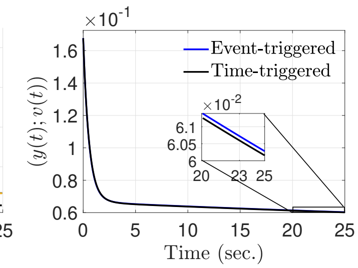
(a) Phase portrait of the event-triggered closed-loop system



(b) Evolution of the error and the threshold



(c) Inter-event times



(d) Performance of time-triggered and event-triggered control

Fig. 1: Simulation results for event-triggered implementation of a controller designed for system (30), with the triggering rule $||e_v|| \geq \sigma(||w|| + y^4)$ for $\sigma = \frac{1}{16}$.

B. Example 2 : Position stabilization of MIP robot

The MIP robot is a four degrees-of-freedom robot with two independently driven wheels and a pendulum-like central body that has unstable pitching motion under the influence of gravity. For event-triggered implementation, we consider the reduced attitude stabilizing controller [5] that asymptotically drives the robot to the origin of the (x, y) plane, while maintaining an upright position.

1) Modelling and Control: The state-space model of the MIP robot was presented in [5] with the state $x \in \mathcal{Q}_c \triangleq (-1, 1)^2 \times S^1 \times \mathbb{R}^3$. The control inputs of the robot are $u = (u_1, u_2)$. It can be verified that the linearized model of the system about $(x; u) = (0; 0)$ satisfies conditions in Assumption 1. A linear state-feedback control law

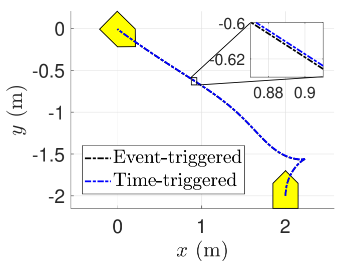
$$u_1 = -K_1[x_2\ x_3\ x_4\ x_5]^\top,\ u_2 = -K_2[x_6\ x_1]^\top \tag{32}$$

where, $K_1 = [k_{1i}]_{i=1,\ldots,4} \in \mathbb{R}^4, K_2 = [k_{2j}]_{j=1,2} \in \mathbb{R}^2$ was proposed in [5] to achieve the control objective of reduced attitude stabilization. The MIP system, following the change of variables $p_1 = x_1$ and $\bar{p}_2 \triangleq (x_2, x_3, x_4, x_5, x_6 + (k_{22}/k_{21})x_1)$, satisfies Assumption 1 and there exists locally, a smooth map $h : \mathbb{R} \to \mathbb{R}^5$ such that $\bar{p}_2 = h(p_1)$ is a center manifold for the MIP system, with $h(p_1) = (c_1p_1^2 + \mathcal{O}(|p_1|^4), \mathcal{O}(|p_1|^4), \mathcal{O}(|p_1|^4), -c_2p_1^2 + \mathcal{O}(|p_1|^4), c_3p_1^3 + \mathcal{O}(|p_1|^4))$ (derived in [5]), where the constants c_1, c_2 and c_3 are functions of the robot parameters and the controller gains k_{ij}.

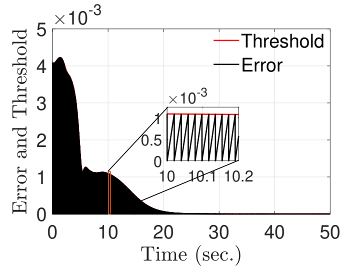
2) Event-triggered implementation: For simulations, the gains in (32) are chosen to be

$$\begin{aligned} u_1 &= 0.1091x_2 + 7.0089x_3 + 1.0014x_4 + 0.4302x_5 \\ u_2 &= -0.1929x_6 - 0.09645x_1. \end{aligned} \tag{33}$$

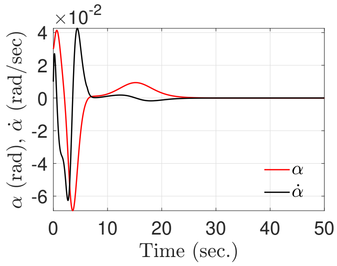
For the controller chosen, it can be checked Assumption 2 is satisfied and we use the relative thresholding $||e_{\bar{p}_2}|| \geq \sigma(||\bar{p}_2 - h^a(p_1)|| + ||p_1||^4)$ from Proposition 4 with $h^a(p_1) \triangleq (c_1p_1^2,\ 0,\ 0,\ -c_2p_1^2,\ c_3p_1^3)$ for event-triggered implementation. The bound on the thresholding parameter σ from (16) is found by solving the Lyapunov equation with $Q = \mathbb{I}_5$ and is found to be $\sigma \leq 10^{-4}$. Under this implementation, asymptotic stability and the non-existence of Zeno behaviour is guaranteed through Proposition 4. With $\sigma = 10^{-4}$, the simulation results of event-triggered position stabilization of the MIP robot are presented in Figures 2 and 3. The MIP robot is initialized at $(x, y, \theta) = (2, 2, \frac{\pi}{2})$ with the pitch upright, that is, $x_3 = x_4 = 0$ and $x_5 = x_6 = 0$. The evolution of the position of the robot is shown in Figure 2a. The robot asymptotically reaches the origin of the (x, y) plane. In Figure 2b, the evolution of the norm of the error $||e_{\bar{p}_2}||$ and the threshold $10^{-4}(||\bar{p}_2 - h^a(p_1)|| + ||p_1||^4)$ are shown. An event occurs when the norm of the error rises from zero to meet the threshold. In Figure 2c, we see that the pitch angle $x_3 = \alpha$ and pitch velocity $x_4 = \dot{\alpha}$ tend to zero asymptotically, as guaranteed by Proposition 4. The evolution of the inter-event times is shown in Figure 2d. The inter-event times are lower bounded by 2.4 ms. Considering ten initial conditions inside a circle of radius 3 m in the (x, y) plane, it is found that the minimum time between two events (MIETs) is 2.4 ms. With MIETs as the sampling time for time-triggered control, the simulations results are shown in Figure 2a. The performance of event-triggered control is a close match to the performance of time-triggered control. However, event-triggered control requires fewer closings of the control-loop than time-triggered control. From Figures 3a,



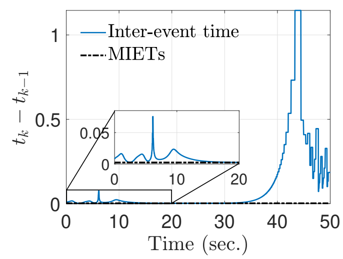
(a) Position (x, y) of the robot.



(b) Evolution of the error and the threshold

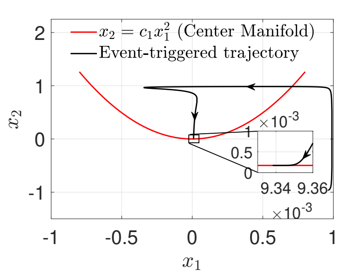


(c) Evolution of α and $\dot{\alpha}$

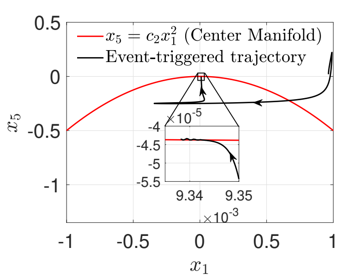


(d) Inter-event times

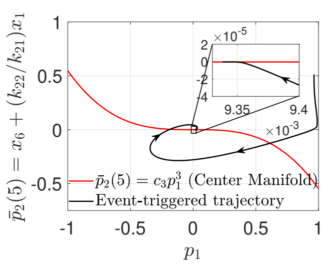
Fig. 2: Simulation results for the event-triggered position stabilization of the MIP robot.



(a) Evolution in (x_1, x_2) plane



(b) Evolution in (x_1, x_5) plane



(c) Evolution in $(x_1, \bar{p}_2(5))$ plane

Fig. 3: The trajectories of the states x_2, x_5 and $\bar{p}_2(5) = x_6 + (k_{22}/k_{21})x_1$ of the event-triggered closed-loop system.

3b and 3c, we see that the trajectories of the event-triggered closed-loop system converge rapidly to the center manifold, while evolving slowly along the center manifold.

VII. Conclusions

In this work, we presented event-triggered implementation of control laws designed for nonlinear systems with center manifolds. The proposed methods ensured Zeno-free local ultimate boudedness of the closed-loop trajectories, and under some assumptions on the controller structure, Zeno-free asymptotic stability of the origin. Systems for which the center manifold is exactly computable were considered first and triggering conditions were presented, the checking of which requires the exact knowledge of the center manifold. Then, we considered systems for which the center manifold can only be approximately computed and showed that the same triggering conditions could be used with the available approximate knowledge of the center manifold. We presented two examples, where the minimum inter-event time from multiple simulations (MIETs) was used as sampling time for time-triggered control and it was found that event-triggered control yields similar performance as time-triggered control but with significantly fewer control updates.

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