E1.216 COMPUTER VISION

LECTURE 09: TWO-VIEW OR EPIPOLAR GEOMETRY

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In this lecture we shall look at **two-view** or **epipolar** geometry

- Epipolar geometry is generalisation of classical stereopsis
- Major breakthrough in understanding multiview geometry

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Epipolar Geometry

Geometry of Two Views

- Epipolar geometry is *intrinsic* projective geometry between two views
- **Crucial feature** : Independent of 3D scene structure
- Depends only on *intrinsic* and *extrinsic* calibration
- Represents a major advancement in geometric understanding
- Can be **computed** using matched points (independent of structure)

Epipolar Geometry

We shall look at

- Properties of epipolar geometry
- Implications for calibrated and uncalibrated cameras
- Special motion cases
- Inference of motion from epipolar geometry
- Estimation of epipolar geometry

Epipolar Geometry

Fundamental Matrix

If 3-D point $\boldsymbol{X}$ is imaged as $\boldsymbol{x}$ and $\boldsymbol{x}'$ in two images, then there is a 3×3 rank-2 matrix $\boldsymbol{F}$ know as the *fundamental matrix* such that

$$\boldsymbol{x}'^T \boldsymbol{F}\boldsymbol{x} = 0$$

We shall look at

- derivations of above *epipolar constraint*
- computation of *fundamental matrix* F

Epipolar Geometry

Two types of matrices

- Two epipolar descriptions - calibrated and uncalibrated
- Uncalibrated is general form of which calibrated is specialisation
- Historically calibrated epipolar geometry was solved first
- Matrix for calibrated case is known as *essential matrix* denoted $\boldsymbol{E}$
- Relationship is $\boldsymbol{x}'^{T}\boldsymbol{E}\boldsymbol{x} = 0$

Two types of matrices

- We shall develop the epipolar relationship in different ways
- $\boldsymbol{F}$ has 7 degrees of freedom
  - 3×3 matrix has 9 degrees of freedom
  - minus 1 for overall scale factor
  - minus 1 for constraint $|\boldsymbol{F}| = 0$
- $\boldsymbol{E}$ has 5 degrees of freedom
  - Calibrated case
  - Rotation and translation are 6 degrees of freedom
  - minus 1 for overall scale factor
- Brief foray into calibrated case, then develop general form
- Will then return to calibrated case

Consider the calibrated case
Wlog we can attach frame of reference to first frame
Note we can only compute *relative* motion between two frames
For 3-D point $\boldsymbol{P} = [\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}]^T$, in projective sense

$$\begin{aligned} \boldsymbol{x} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} &= \frac{1}{\boldsymbol{Z}} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{Z} \end{bmatrix} \\ \Rightarrow \boldsymbol{x} &= \boldsymbol{P} \end{aligned}$$

Epipolar Geometry

Now let us rotate the camera by $\boldsymbol{R}$ and translate by $\boldsymbol{T}$
3-D point in new co-ordinate system has position

$$\boldsymbol{P}^{'} = \boldsymbol{R}\begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{Z} \end{bmatrix} + \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix}$$

Projectively, the image co-ordinates in the second image is given by

$$\begin{aligned} \boldsymbol{x}^{'} &= \lambda \boldsymbol{P}^{'} \\ &= \lambda \boldsymbol{R}\begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{Z} \end{bmatrix} + \lambda \boldsymbol{T} \end{aligned}$$

We further develop this relationship

$$
\begin{aligned}
\boldsymbol{x}^{'} &= \lambda \boldsymbol{P}^{'} \\
&= \lambda \boldsymbol{R} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{Z} \end{bmatrix} + \lambda \boldsymbol{T} \\
&= \lambda \boldsymbol{Z} \boldsymbol{R} \boldsymbol{x} + \lambda \boldsymbol{T}
\end{aligned}
$$

We have

$$\boldsymbol{x}^{'} = \lambda \boldsymbol{Z}\boldsymbol{R}\boldsymbol{x} + \lambda \boldsymbol{T}$$

Now taking cross-product with $\boldsymbol{T}$ on either side we eliminate $\boldsymbol{T}$ term on rhs

$$\boldsymbol{T} \times \boldsymbol{x}^{'} = \boldsymbol{T} \times \lambda \boldsymbol{Z}\boldsymbol{R}\boldsymbol{x}$$

Given

$$\boldsymbol{T} \times \boldsymbol{x}^{'} = \boldsymbol{T} \times \lambda \boldsymbol{Z} \boldsymbol{R} \boldsymbol{x}$$

obviously the following dot-product relationship is true

$$\boldsymbol{x}^{'T} \boldsymbol{T} \times \boldsymbol{x}^{'} = \boldsymbol{x}^{'T} \boldsymbol{T} \times \lambda \boldsymbol{Z} \boldsymbol{R} \boldsymbol{x} = 0$$

Note that $\boldsymbol{a} \times \boldsymbol{b}$ can be written as

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \begin{bmatrix} a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ &= [\boldsymbol{a}]_\times \boldsymbol{b} \end{aligned}$$

- Represent cross-product in $\boldsymbol{A}\boldsymbol{x}$ form
- $[.]_\times$ is a *skew-symmetric* matrix

Given this we now have the relationship for
epipolar geometry

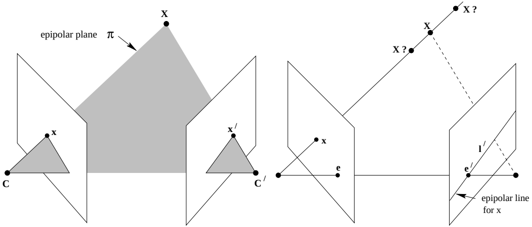
$$\begin{aligned} \boldsymbol{x}'^{T} [\boldsymbol{T}]_{\times} \boldsymbol{R}\boldsymbol{x} &= 0 \\ \Rightarrow \boldsymbol{x}'^{T} \boldsymbol{E}\boldsymbol{x} &= 0 \end{aligned}$$

Essential matrix $\boldsymbol{E}$ has form

$$\boldsymbol{E} = [\boldsymbol{T}]_{\times} \boldsymbol{R}$$

- Note the relationship is *independent* of structure
- This was an algebraic derivation
- Now we consider the problem geometrically

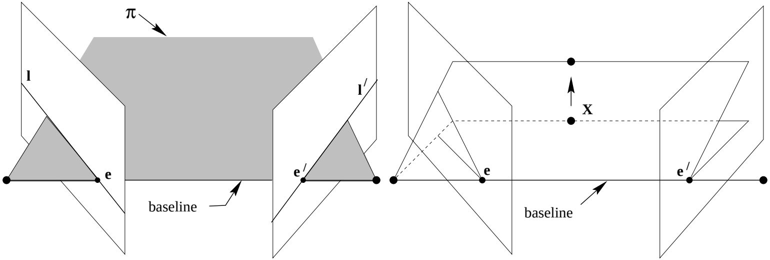
Epipolar Geometry



What kind of geometric relations obtain ?

- 3-D point $\boldsymbol{X}$, image points $\boldsymbol{x}$ and $\boldsymbol{x}^{'}$ are coplanar
- Ray connecting $\boldsymbol{X}$ and $\boldsymbol{x}$ intersects second image in line
- As seen earlier, search space is now a line
- Corresponding epipolar lines for points $\boldsymbol{x}$ and $\boldsymbol{x}^{'}$ in other image
- $\boldsymbol{C}$ and $\boldsymbol{C}^{'}$ are two camera centres
- $\boldsymbol{C}$ and $\boldsymbol{C}^{'}$ always contained in π
- Line connecting two centres is called **baseline**
- Nomenclature of **baseline** is from stereo setting

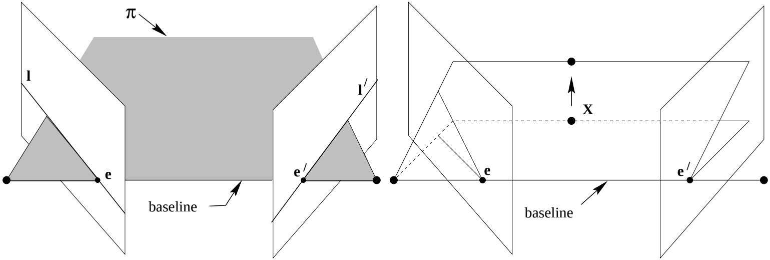
Epipolar Geometry



Epipole

- Point of intersection of baseline with image plane
- Denoted $\boldsymbol{e}$ and $\boldsymbol{e}^{\prime}$ respectively
- Equivalently image of other camera centre on image plane

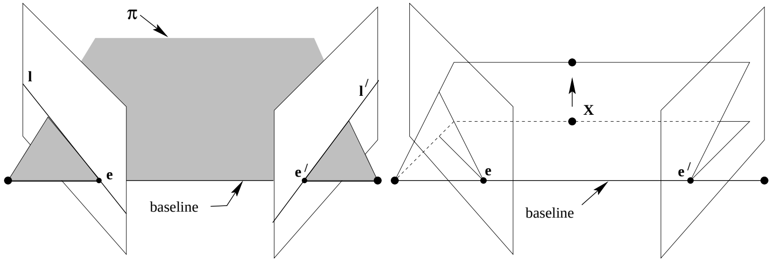
Epipolar Geometry



Epipolar Plane

- Plane containing baseline
- One parameter family (pencil) of epipolar planes
- Each plane allows for *transfer* of points

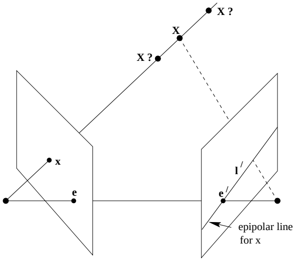
Epipolar Geometry



Epipolar Line

- Intersection of epipolar plane with image plane
- All epipolar lines intersect at epipole (why ?)
- Each line defines matching search space in image plane
- Generalisation from earlier *stereo* setting

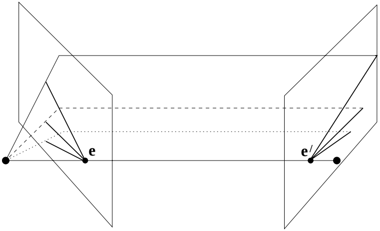
Epipolar Geometry

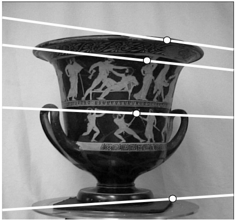


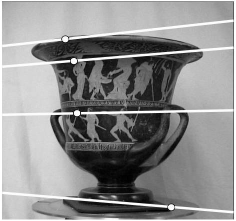
Fundamental Matrix F

- $\forall \boldsymbol{x} \ \exists \boldsymbol{l}'$ in other image
- Correspondence $\boldsymbol{x}'$ must lie on $\boldsymbol{l}'$
- Epipolar line is projection in second image of ray from $\boldsymbol{x}$ through $\boldsymbol{C}$
- This is a mapping $\boldsymbol{x} \mapsto \boldsymbol{l}'$
- Mapping is mediated through the fundamental matrix

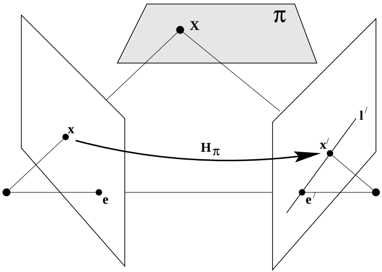
Epipolar Geometry







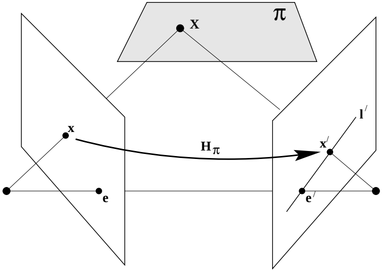
Epipolar Geometry



Geometric Derivation

- Consider plane π not containing either camera centre
- Ray through $\boldsymbol{x}$ and $\boldsymbol{C}$ meets π in $\boldsymbol{X}$
- $\boldsymbol{X}$ gets projected to $\boldsymbol{x}^{'}$ in second image
- *Transfer* via plane π
- Since $\boldsymbol{X}$ is on ray through $\boldsymbol{x}$, $\boldsymbol{x}^{'}$ must lie on $\boldsymbol{l}^{'}$
- $\boldsymbol{x}, \boldsymbol{x}^{'}$ and $\boldsymbol{X}$ all lie on plane

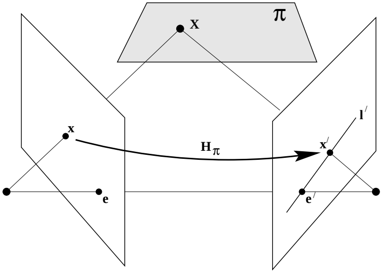
Epipolar Geometry



Geometric Derivation

- $\boldsymbol{x}, \boldsymbol{x}^{'}$ and $\boldsymbol{X}$ all lie on plane
- Set of $\boldsymbol{x}_i$ and corresponding $\boldsymbol{x}_i^{'}$ are projectively equivalent
- This is true because all $\boldsymbol{X}_i$ lie on π
- $\therefore \exists H_\pi$ mapping each $\boldsymbol{x}_i$ to $\boldsymbol{x}_i^{'}$

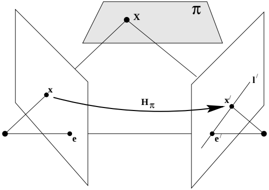
Epipolar Geometry



Constructing the epipolar line

- Epipolar line $\boldsymbol{l}'$ passes through $\boldsymbol{x}'$ and $\boldsymbol{e}'$
- $\boldsymbol{l}' = \boldsymbol{e}' \times \boldsymbol{x}' = \left[\boldsymbol{e}'\right]_{\times} \boldsymbol{x}'$
- Also $\boldsymbol{x}' = H_\pi \boldsymbol{x}$

Epipolar Geometry



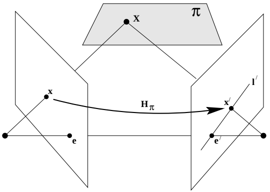
Constructing the epipolar line

- Epipolar line $\boldsymbol{l}^{'}$ passes through $\boldsymbol{x}^{'}$ and $\boldsymbol{e}^{'}$
- $\boldsymbol{l}^{'} = \boldsymbol{e}^{'} \times \boldsymbol{x}^{'} = \left[\boldsymbol{e}^{'}\right]_{\times} \boldsymbol{x}^{'}$
- Also $\boldsymbol{x}^{'} = H_{\pi}\boldsymbol{x}$ therefore, the epipolar line is described by

$$\boldsymbol{l}^{'} = \left[\boldsymbol{e}^{'}\right]_{\times} H_{\pi}\boldsymbol{x} = \boldsymbol{F}\boldsymbol{x}$$

$\boldsymbol{F}$ is known as the **fundamental matrix**

Epipolar Geometry



The fundamental matrix $\boldsymbol{F}$ *may be written as* $\boldsymbol{F} = \left[\boldsymbol{e}'\right]_{\times} H_{\pi}$*, where* H_{π} *is the transfer mapping from one image to another via any plane* π*. Furthermore, since* $\left[\boldsymbol{e}'\right]_{\times}$ *has rank 2 and* H_{π} *rank 3,* $\boldsymbol{F}$ *is a matrix of rank 2*

Epipolar Geometry

Geometric interpretation

- $\boldsymbol{F}$ represents mapping from 2-D projective plane $\mathbb{P}^2$ of first image to pencil of epipolar lines through epipole $\boldsymbol{e}'$
- Mapping from 2-D onto 1-D projective space
- Must have rank 2
- Plane π is *virtual*, i.e. only conceptual

Epipolar Geometry

We can also derive this relationship via the camera projection matrices
Consider the projection equation $P\boldsymbol{X} = \boldsymbol{x}$ and now consider the back-projection from $\boldsymbol{x}$ in given image
Since back-projection is through camera centre, we have

$$\boldsymbol{X}(\lambda) = P^{\dagger}\boldsymbol{x} + \lambda\boldsymbol{C}$$

- $P^{\dagger}$ is pseudo-inverse of P, i.e. $P^{\dagger}P = I$
- $\boldsymbol{C}$ is null-vector, i.e. camera centre satisfies $P\boldsymbol{C} = 0$
- Consider $\lambda = 0$ and $\lambda = \infty$
- Corresponding points are $P^{\dagger}\boldsymbol{x}$ and $\boldsymbol{C}$
- Their projections onto second image are $P^{'}P^{\dagger}\boldsymbol{x}$ and $P^{'}\boldsymbol{C}$
- Epipolar line in second image is $\boldsymbol{l}^{'} = (P^{'}\boldsymbol{C}) \times (P^{'}P^{\dagger}\boldsymbol{x})$
- $P^{'}\boldsymbol{C}$ is epipole in second image $\boldsymbol{e}^{'}$
- Results in relationship

$$\boldsymbol{l}^{'} = \left[\boldsymbol{e}^{'}\right]_{\times}(P^{'}P^{\dagger})\boldsymbol{x} = \boldsymbol{F}\boldsymbol{x}$$

Epipolar Geometry

- Epipolar line in second image is $\boldsymbol{l}^{'} = (P^{'}\boldsymbol{C}) \times (P^{'}P^{\dagger}\boldsymbol{x})$
- $P^{'}\boldsymbol{C}$ is epipole in second image $\boldsymbol{e}^{'}$
- Results in relationship

$$\boldsymbol{l}^{'} = \left[\boldsymbol{e}^{'}\right]_{\times}(P^{'}P^{\dagger})\boldsymbol{x} = \boldsymbol{F}\boldsymbol{x}$$

- $F = \left[\boldsymbol{e}^{'}\right]_{\times}(P^{'}P^{\dagger})$
- H_{π} now has an explicit form, i.e. $H_{\pi} = P^{'}P^{\dagger}$
- Derivation only works for different camera centres

Epipolar Geometry

Consider the two image scenario where projection matrices are

$$
\begin{aligned}
P &= K\left[I|\mathbf{0}\right] \quad P^{'} = K^{'}\left[\boldsymbol{R}|\boldsymbol{t}\right] \\
\Rightarrow P^{\dagger} &= \left[\begin{array}{c} K^{-1} \\ \mathbf{0}^{T} \end{array}\right] \quad \boldsymbol{C} = \left(\begin{array}{c} \mathbf{0} \\ 1 \end{array}\right) \\
\Rightarrow F &= \left[P^{'}\boldsymbol{C}\right]_{\times} P^{'}P^{\dagger} \\
&= \left[K^{'}\boldsymbol{t}\right]_{\times} K^{'}\boldsymbol{R}K^{-1} = \underline{K^{'-T}[\boldsymbol{t}]_{\times}\boldsymbol{R}K^{-1}} \\
&= K^{'-T}\boldsymbol{R}[\boldsymbol{R}\boldsymbol{t}]_{\times}K^{-1} = K^{'-T}\boldsymbol{R}K^{T}\left[K\boldsymbol{R}^{T}\boldsymbol{t}\right]_{\times}
\end{aligned}
$$

Where are the epipoles ?

$$\begin{aligned} \boldsymbol{e} &= P\begin{pmatrix} -\boldsymbol{R}^T\boldsymbol{t} \\ 1 \end{pmatrix} = K\boldsymbol{R}^T\boldsymbol{t} \\ \boldsymbol{e}^{'} &= P^{'}\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} = K^{'}\boldsymbol{t} \end{aligned}$$

Also, the most useful form of $\boldsymbol{F}$ is

$$\boldsymbol{F} = K^{'-T}[\boldsymbol{t}]_{\times}\boldsymbol{R}K^{-1}$$

Epipolar Geometry

Consider the representation of the fundamental matrix

$$\begin{aligned} \boldsymbol{F} &= K'^{-T}[\boldsymbol{t}]_\times \boldsymbol{R} K^{-1} \\ \Rightarrow \boldsymbol{x}'^T \boldsymbol{F} \boldsymbol{x} &= 0 \\ \Rightarrow \boldsymbol{x}'^T K'^{-T}[\boldsymbol{t}]_\times \boldsymbol{R} K^{-1} \boldsymbol{x} &= 0 \end{aligned}$$

This can be interpreted as

$$\underbrace{\boldsymbol{x}'^T K'^{-T}}[\boldsymbol{t}]_\times \boldsymbol{R} \overbrace{K^{-1}\boldsymbol{x}} = 0$$

- Terms denoted are the calibrated image points
- Central term is the *essential matrix*

Epipolar Geometry

$$\boldsymbol{x}^{'T} \boldsymbol{F} \boldsymbol{x} = 0$$

Properties : Transpose

- F for pair $(P, P^{'})$ implies F^{T} for $(P^{'}, P)$

Epipolar Geometry

$$\boldsymbol{x}^{'T}\boldsymbol{F}\boldsymbol{x} = 0$$

Properties : Epipolar Lines

- For $\boldsymbol{x}$ corresponding epipolar line is $\boldsymbol{l}^{'} = \boldsymbol{F}\boldsymbol{x}$
- For $\boldsymbol{x}^{'}$ corresponding epipolar line is $\boldsymbol{l} = \boldsymbol{F}^{T}\boldsymbol{x}^{'}$
- Further for corresponding epipolar lines $\boldsymbol{l}$ and $\boldsymbol{l}^{'}$
  $\boldsymbol{l}^{'} = \boldsymbol{F}[\boldsymbol{e}]_{\times}\boldsymbol{l}; \quad \boldsymbol{l} = \boldsymbol{F}^{T}\left[\boldsymbol{e}^{'}\right]_{\times}\boldsymbol{l}^{'}$

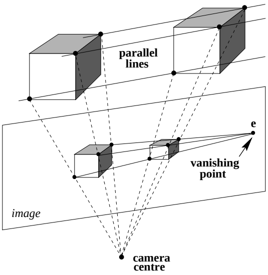
Epipolar Geometry

$$\boldsymbol{x}'^{T}\boldsymbol{F}\boldsymbol{x} = 0$$

Properties : Epipoles

- Any point $\boldsymbol{x}$ other than $\boldsymbol{e}$, $\boldsymbol{l}' = \boldsymbol{F}\boldsymbol{x}$ contains $\boldsymbol{e}'$
- Implies $\boldsymbol{e}'^{T}\boldsymbol{F}\boldsymbol{x} = (\boldsymbol{e}'^{T}\boldsymbol{F})\boldsymbol{x} = 0, \forall \boldsymbol{x}$
- $\therefore \boldsymbol{e}'^{T}\boldsymbol{F} = 0, \forall \boldsymbol{x}$, i.e. $\boldsymbol{e}'$ is left null-vector of $\boldsymbol{F}$
- Similarly, $\boldsymbol{F}\boldsymbol{e} = 0$

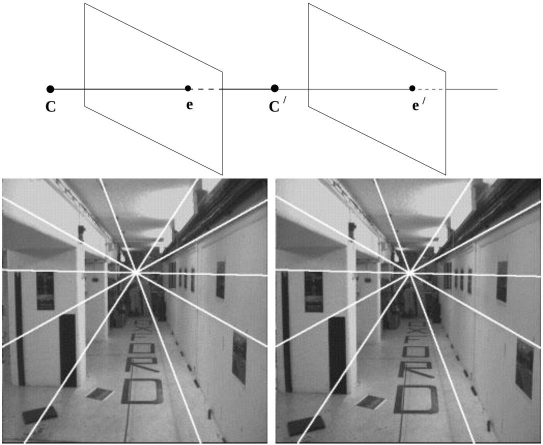
Epipolar Geometry



Special Motions : Pure Translation

- Consider pure translation $\boldsymbol{t}$
- Points in 3-D move on lines parallel to $\boldsymbol{t}$
- Image intersection of these parallel lines is the vanishing point $\boldsymbol{v}$
- $\boldsymbol{v}$ in the direction of $\boldsymbol{t}$
- $\boldsymbol{v}$ is the epipole of both views
- Imaged parallel lines are epipolar lines

Epipolar Geometry



Special Motions : Pure Translation

- Assuming camera parameters do not change
- $P = \boldsymbol{K}\left[\boldsymbol{I}|\boldsymbol{0}\right]; \quad P^{'} = \boldsymbol{K}\left[\boldsymbol{I}|\boldsymbol{t}\right]$
- $\therefore \boldsymbol{F} = \left[\boldsymbol{e}^{'}\right]_{\times} \boldsymbol{K}\boldsymbol{K}^{-1} = \left[\boldsymbol{e}^{'}\right]_{\times}$
- Ideal rectified stereo is a special case of this

Epipolar Geometry

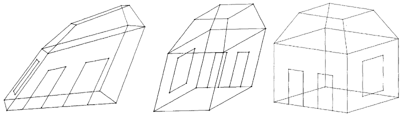


Figure 6.10. Illustration of the stratified approach: projective structure $\boldsymbol{X}_p$, affine structure $\boldsymbol{X}_a$, and Euclidean structure $\boldsymbol{X}_e$ obtained in different stages of reconstruction.

Projective Ambiguity

- Camera pairs $(P, P^{'})$ and $(PH, P^{'}H)$ are equivalent
- H is a 4×4 projective transformation
- Recovery of camera pairs given F matrix
- Notice that $P\boldsymbol{X} = (PH)(H^{-1}\boldsymbol{X})$
- Basic asymmetry :
  - Camera pairs determine F uniquely
  - F does not do that for camera pairs
  - Ambiguity of “projective basis”

Canonical Form of Projective Matrices

- Given above projective ambiguity, need to fix it
- Note that we always only measure "relative" motion
- Fix reference frame to first image, i.e. $P = [\boldsymbol{I}|\boldsymbol{0}]$
- Consequently $P^{'} = [\boldsymbol{M}|\boldsymbol{m}]$
- Here $\boldsymbol{F} = [\boldsymbol{m}]_{\times}\boldsymbol{M}$

Now we **specialise** to the *essential matrix*, i.e. calibrated cameras

Normalised coordinates

- Let $P = \boldsymbol{K}\,[\boldsymbol{R}|\boldsymbol{t}]$
- Assume we know or can estimate $\boldsymbol{K}$
- Can apply $\boldsymbol{K}^{-1}$ to image coordinates $\boldsymbol{x}$
- Normalised coordinates $\hat{\boldsymbol{x}} = \boldsymbol{K}^{-1}\boldsymbol{x}$
- $\hat{\boldsymbol{x}} = [\boldsymbol{R}|\boldsymbol{t}]\,\boldsymbol{X}$

Epipolar Geometry

Consider camera pairs $P = [\boldsymbol{I}|\boldsymbol{0}]$ and $P^{'} = [\boldsymbol{R}|\boldsymbol{t}]$
Essential matrix is given by

$$\begin{aligned} \boldsymbol{E} = [\boldsymbol{t}]_\times \boldsymbol{R} &= \boldsymbol{R}\left[\boldsymbol{R}^T \boldsymbol{t}\right]_\times \\ \hat{\boldsymbol{x}}^{'T} \boldsymbol{E} \hat{\boldsymbol{x}} &= 0 \end{aligned}$$

As shown earlier, we have

$$\boldsymbol{E} = \boldsymbol{K}^{'T} \boldsymbol{F} \boldsymbol{K}$$

Essential Matrix properties

- Has fewer degrees of freedom than the fundamental matrix
- Defined by rotation and translation, i.e. six degrees of freedom
- However there's an overall scale ambiguity
- The *essential matrix* has 5 degrees of freedom
- Implies a translation scale ambiguity
- We can only solve for heading direction and not actual translation

Epipolar Geometry

A 3×3 matrix is an essential matrix if and only if two if its singular values are equal, and the third is zero

Essential Matrix properties

- Rank-2 constraint implies third singular value is zero
- Canonical form is singular values of $(1, 1, 0)$
- Alternate form: $(\boldsymbol{E}\boldsymbol{E}^T)\boldsymbol{E} - \frac{1}{2}tr(\boldsymbol{E}\boldsymbol{E}^T)\boldsymbol{E} = \mathbf{0}$

Epipolar Geometry

Proof Consider the decomposition $\boldsymbol{E} = [\boldsymbol{t}]_\times \boldsymbol{R} = \boldsymbol{S}\boldsymbol{R}$

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- W is orthogonal
- Z is skew-symmetric
- Skew-symmetric $\boldsymbol{S}$ can always be written as $\boldsymbol{S} = k\boldsymbol{U}Z\boldsymbol{U}^T$
- $\boldsymbol{U}$ is orthogonal
- Z = diag(1,1,0)W upto scale
- Implies $\boldsymbol{S} = \boldsymbol{U} diag(1,1,0) W \boldsymbol{U}^T$
- $\boldsymbol{E} = \boldsymbol{S}\boldsymbol{R} = \boldsymbol{U} diag(1,1,0) W \boldsymbol{U}^T \boldsymbol{R}$
- Above is a singular value decomposition of $\boldsymbol{E}$ (Q.E.D.)

Epipolar Geometry

Decomposing Essential Matrix

- SVD of $\boldsymbol{E}$ is $\boldsymbol{U}\,diag(1,1,0)\,\boldsymbol{V}^T$
- Ignoring signs, there are two factorisations
  - $\boldsymbol{S} = \boldsymbol{U}Z\boldsymbol{U}^T$ and $\boldsymbol{R} = \boldsymbol{U}W\boldsymbol{V}^T$
  - $\boldsymbol{S} = \boldsymbol{U}Z\boldsymbol{U}^T$ and $\boldsymbol{R} = \boldsymbol{U}W^T\boldsymbol{V}^T$
- $\boldsymbol{t}$ is (upto scale) the last column of $\boldsymbol{U}$
- Sign of $\boldsymbol{t}$ is ambiguous
- Results in 4 possible decomposition pairs
- Need to verify *depth positivity* of one point to disambiguate
- *Depth positivity* will give an unambiguous solution

We now shift gears and consider computation of the Fundamental Matrix

Epipolar Geometry Computation

$$\boldsymbol{x}'^T \boldsymbol{F} \boldsymbol{x} = 0$$

Linear Estimation of $\boldsymbol{F}$

- Above equation can solve for $\boldsymbol{F}$ given enough $(\boldsymbol{x}, \boldsymbol{x}')$ pairs
- A minimum of 7 matched points are required
- Denote $\boldsymbol{x} = (x, y, 1)^T$ and $\boldsymbol{x}' = (x', y', 1)^T$
- Can write the *bilinear form* as linear equation in entries of $\boldsymbol{F}$

Epipolar Geometry Computation

$$\begin{aligned}
\boldsymbol{x}^{'T}\boldsymbol{F}\boldsymbol{x} &= 0 \\
\Rightarrow \begin{bmatrix} x^{'} & y^{'} & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} &= 0 \\
\Rightarrow \begin{pmatrix} x^{'}x & x^{'}y & x^{'} & y^{'}x & y^{'}y & y^{'} & x & y & 1 \end{pmatrix} \boldsymbol{f} &= 0 \\
\Rightarrow \left(\boldsymbol{x}^{'} \otimes \boldsymbol{x}\right)^{T} \boldsymbol{f} &= 0
\end{aligned}$$

Epipolar Geometry Computation

$$\boldsymbol{A}\boldsymbol{f} = \begin{bmatrix} x'_1 x_1 & x'_1 y_1 & x'_1 & y'_1 x_1 & y'_1 y_1 & y'_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_n x_n & x'_n y_n & x'_n & y'_n x_n & y'_n y_n & y'_n & x_n & y_n & 1 \end{bmatrix} \boldsymbol{f} = 0$$

Eight-Point Algorithm

- Homogeneous set of equations
- $\boldsymbol{f}$ determined only upto scale
- For solution, $\boldsymbol{A}$ must be of rank at most 8
- For rank $= 8$, solution is exact (upto scale)
- Solution is the right null-space of $\boldsymbol{A}$
- Solved using SVD
- Very popular and useful algorithm

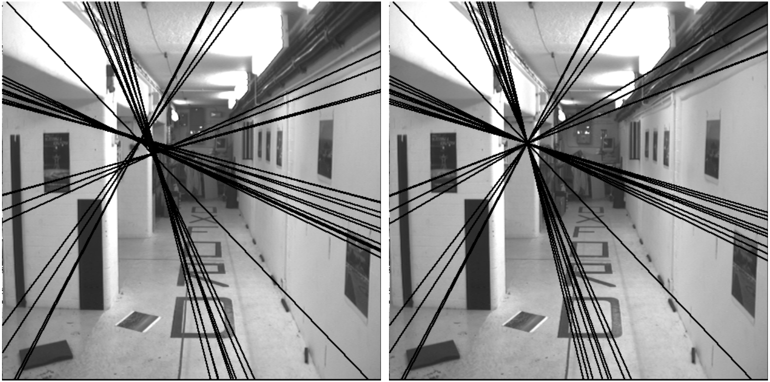
Epipolar Geometry Computation

$$\boldsymbol{A}\boldsymbol{f} = \begin{bmatrix} x'_1 x_1 & x'_1 y_1 & x'_1 & y'_1 x_1 & y'_1 y_1 & y'_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_n x_n & x'_n y_n & x'_n & y'_n x_n & y'_n y_n & y'_n & x_n & y_n & 1 \end{bmatrix} \boldsymbol{f} = 0$$

Eight-Point Algorithm

- When data is noisy, $\boldsymbol{A}$ is full rank
- Use $\boldsymbol{A}$ to find least squares solution
- Lsq solution is smallest right-singular vector of $\boldsymbol{A}$
- Equivalent to minimising *algebraic error* $||\boldsymbol{A}\boldsymbol{f}||$

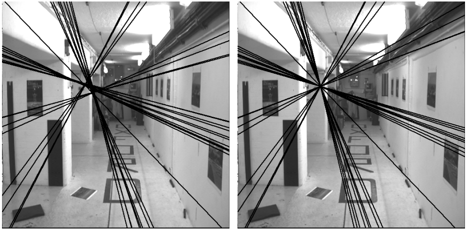
Epipolar Geometry Computation



Linear Estimation

- $\boldsymbol{F}$ is a rank-2 matrix
- Rank constraint not enforced for linear estimate
- Leads to violation of the epipolar geometry properties
- Need to enforce the rank-2 constraint on $\boldsymbol{F}$

Epipolar Geometry Computation



Enforcing rank-2 constraint

- Project $\boldsymbol{F}$ to closest rank-2 matrix, say $\boldsymbol{F}^{'}$
- Distance is the Frobenius norm $||\boldsymbol{F} - \boldsymbol{F}^{'}||$
- Minimise Frobenius distance using SVD
- Let $\boldsymbol{F} = \boldsymbol{U} diag(r, s, t) \boldsymbol{V}^T$
- Set t = 0, i.e. $\boldsymbol{F}^{'} = \boldsymbol{U} diag(r, s, 0) \boldsymbol{V}^T$
- Need to enforce the rank-2 constraint on $\boldsymbol{F}$
- This is guaranteed to minimise required measure

Epipolar Geometry Computation

250906.36	183269.57	921.81	200931.10	146766.13	738.21	272.19	198.81	1.00
2692.28	131633.03	176.27	6196.73	302975.59	405.71	15.27	746.79	1.00
416374.23	871684.30	935.47	408110.89	854384.92	916.90	445.10	931.81	1.00
191183.60	171759.40	410.27	416435.62	374125.90	893.65	465.99	418.65	1.00
48988.86	30401.76	57.89	298604.57	185309.58	352.87	846.22	525.15	1.00
164786.04	546559.67	813.17	1998.37	6628.15	9.86	202.65	672.14	1.00
116407.01	2727.75	138.89	169941.27	3982.21	202.77	838.12	19.64	1.00
135384.58	75411.13	198.72	411350.03	229127.78	603.79	681.28	379.48	1.00

Typical $\boldsymbol{A}$ matrix

Normalised Eight Point Algorithm

- Original 8-pt algorithm due to Longuet-Higgins (1981)
- Need to take special care when data is noisy
- Note the linear row-term of $\boldsymbol{A}$ is $\left(\boldsymbol{x}' \otimes \boldsymbol{x}\right)^T$
- Each term in kronecker product is of different scale
- Noise is scaled differently in each term

Table taken from slides of Marc Pollefeys

Normalised Eight Point Algorithm

- Original 8-pt algorithm due to Longuet-Higgins (1981)
- Need to take special care when data is noisy
- Note the linear row-term of $\boldsymbol{A}$ is $\left(\boldsymbol{x}^{'} \otimes \boldsymbol{x}\right)^{T}$
- Each term in kronecker product is of different scale
- Noise is scaled differently in each term
- **Linear Algebra view** : Poorly conditioned computation
- **Statistical Estimation view** : Non-white covariance of noise (data dependent)
- Need to scale computation (i.e. whiten) to make it useful
- Hartley's paper : "In Defence of the Eight-Point Algorithm"

Normalised Eight Point Algorithm

- Need to scale data appropriate to improve conditioning
- Transformed data should be close to “whitened” data
- Done by translating and scaling
- **Translation** : Remove centroid of data
- **Scaling** : Make RMS distance from origin equal to $\sqrt{2}$
- Apply transforms $\boldsymbol{T}$ and $\boldsymbol{T}^{'}$ to sets $\boldsymbol{x}$ and $\boldsymbol{x}^{'}$
- Compute linear estimate of fundamental matrix $\boldsymbol{F}$
- Enforce rank-2 constraint
- Map back to image co-ordinate by putting back $\boldsymbol{T}$ and $\boldsymbol{T}^{'}$

Epipolar Geometry Computation

Objective

Given $n \geq 8$ image point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}$, determine the fundamental matrix F such that $\mathbf{x}'^{\top}_i \mathrm{F} \mathbf{x}_i = 0$.

Algorithm

(i) **Normalization:** Transform the image coordinates according to $\hat{\mathbf{x}}_i = \mathrm{T}\mathbf{x}_i$ and $\hat{\mathbf{x}}'_i = \mathrm{T}'\mathbf{x}'_i$, where T and T′ are normalizing transformations consisting of a translation and scaling.

(ii) Find the fundamental matrix $\hat{\mathrm{F}}'$ corresponding to the matches $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$ by

  (a) **Linear solution:** Determine $\hat{\mathrm{F}}$ from the singular vector corresponding to the smallest singular value of $\hat{\mathrm{A}}$, where $\hat{\mathrm{A}}$ is composed from the matches $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$ as defined in (11.3).

  (b) **Constraint enforcement:** Replace $\hat{\mathrm{F}}$ by $\hat{\mathrm{F}}'$ such that $\det \hat{\mathrm{F}}' = 0$ using the SVD (see section 11.1.1).

(iii) **Denormalization:** Set $\mathrm{F} = \mathrm{T}'^{\top}\hat{\mathrm{F}}'\mathrm{T}$. Matrix F is the fundamental matrix corresponding to the original data $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$.

Algorithm 11.1. *The normalized 8-point algorithm for* F.

Epipolar Geometry Computation

Estimation

- Normalised Eight Point algorithm works well, but not optimal
- Optimal estimate would require direct enforcement of rank-2 constraint
- Results in non-linear iterative minimisation of algebraic error $||\boldsymbol{A}\boldsymbol{f}||$
- Algebraic error does not have real geometric significance
- Can define geometric error terms and minimise non-linearly
- Geometric error term will be measurable in the image plane

Seven Point Algorithm

- Fundamental matrix $\boldsymbol{F}$ has seven degrees of freedom
- Each point correspondence pair provides one constraint
- Hence should be able to solve for $\boldsymbol{F}$ using seven points
- Null space is now two-dimensional, say spanned by $\boldsymbol{F}_1$ an $\boldsymbol{F}_2$
- Rank constraint is $|\boldsymbol{F}_1 + \lambda \boldsymbol{F}_2| = 0$
- Cubic equation in λ, solve analytically
- Either one or three real solutions
- May need extra points for verification
- **Estimation:** Use same SVD approach as for 8-point algorithm

Epipolar Geometry Computation

Minimise

$$\sum_i d(\boldsymbol{x}_i, \hat{\boldsymbol{x}}_i)^2 + d(\boldsymbol{x}_i{}', \hat{\boldsymbol{x}}_i{}')^2$$

Geometric Error Minimisation

- $\boldsymbol{x}_i \leftrightarrow \boldsymbol{x}_i{}'$ are measured correspondences
- $\hat{\boldsymbol{x}}_i$ and $\hat{\boldsymbol{x}}_i{}'$ are "true" matches satisfying epipolar constraints
- Assume camera matrices $P = [\boldsymbol{I}|\boldsymbol{0}]$ and $P' = [M|\boldsymbol{t}]$
- Vary P' and $\boldsymbol{X}_i$ and measure above "reprojection" error
- After minimisation, compute $\boldsymbol{F} = [\boldsymbol{t}]_\times M$
- Note that rank-2 constraint is automatically enforced here
- This is the optimal "Gold Standard" solution
- Non-linear optimisation is expensive and needs good initialisation
- Cost substantially reduced using sparse Levenberg-Marquardt

Epipolar Geometry Computation

<u>Objective</u>

Given $n \geq 8$ image point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}$, determine the Maximum Likelihood estimate $\hat{\mathtt{F}}$ of the fundamental matrix.

The MLE involves also solving for a set of subsidiary point correspondences $\{\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i\}$, such that $\hat{\mathbf{x}}'^{\mathsf{T}}_i \hat{\mathtt{F}} \hat{\mathbf{x}}_i = 0$, and which minimizes

$$\sum_i d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2 + d(\mathbf{x}'_i, \hat{\mathbf{x}}'_i)^2.$$

<u>Algorithm</u>

(i) Compute an initial rank 2 estimate of $\hat{\mathtt{F}}$ using a linear algorithm such as algorithm 11.1.

(ii) Compute an initial estimate of the subsidiary variables $\{\hat{\mathbf{x}}_i, \hat{\mathbf{x}}'_i\}$ as follows:

- (a) Choose camera matrices $\mathtt{P} = [\mathtt{I} \mid \mathbf{0}]$ and $\mathtt{P}' = [[\mathbf{e}']_\times \hat{\mathtt{F}} \mid \mathbf{e}']$, where $\mathbf{e}'$ is obtained from $\hat{\mathtt{F}}$.
- (b) From the correspondence $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$ and $\hat{\mathtt{F}}$ determine an estimate of $\widehat{\mathbf{X}}_i$ using the triangulation method of chapter 12.
- (c) The correspondence consistent with $\hat{\mathtt{F}}$ is obtained as $\hat{\mathbf{x}}_i = \mathtt{P}\widehat{\mathbf{X}}_i$, $\hat{\mathbf{x}}'_i = \mathtt{P}'\widehat{\mathbf{X}}_i$.

(iii) Minimize the cost

$$\sum_i d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2 + d(\mathbf{x}'_i, \hat{\mathbf{x}}'_i)^2$$

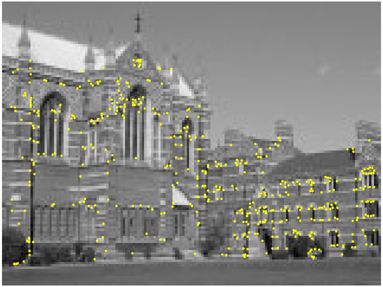
over $\hat{\mathtt{F}}$ and $\widehat{\mathbf{X}}_i$, $i = 1, \ldots, n$. The cost is minimized using the Levenberg–Marquardt algorithm over $3n + 12$ variables: $3n$ for the n 3D points $\widehat{\mathbf{X}}_i$, and 12 for the camera matrix $\mathtt{P}' = [\mathtt{M} \mid \mathbf{t}]$, with $\hat{\mathtt{F}} = [\mathbf{t}]_\times \mathtt{M}$, and $\hat{\mathbf{x}}_i = \mathtt{P}\widehat{\mathbf{X}}_i$, $\hat{\mathbf{x}}'_i = \mathtt{P}'\widehat{\mathbf{X}}_i$.

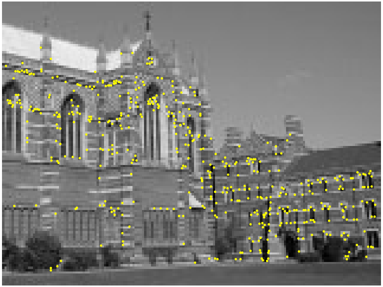
Algorithm 11.3. *The Gold Standard algorithm for estimating* F *from image correspondences.*

Automatic Computation of F

- Input is only two images
- Compute **interest points**
- **Match** points across two images using local neighbourhood
- Run **RANSAC** robust estimator for removing outliers
- Estimate $\boldsymbol{F}$ matrix using inliers and improve using *non-linear estimate*
- Use estimate of $\boldsymbol{F}$ to **guide matching** by search in a band around epipolar lines
- Iterate till the number of correspondences obtained is stable

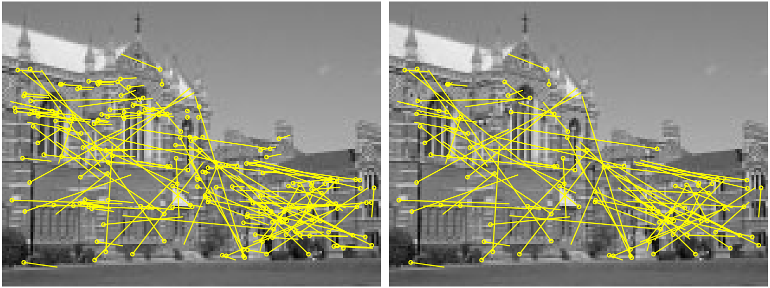
Left and right images





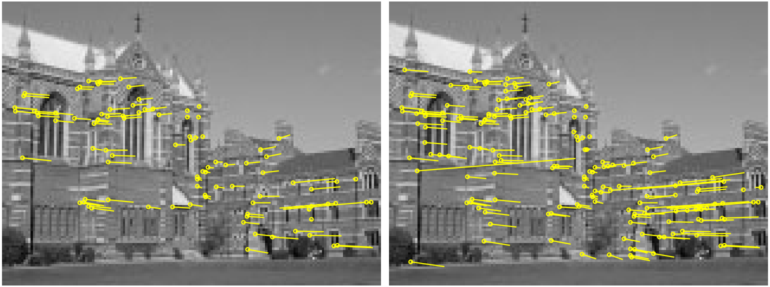
Detected Corners

Epipolar Geometry Computation



Putative matches that have many wrong ones!

Epipolar Geometry Computation



Inliers and final set of matches (including a few wrong ones)